CHAPTER 2

CONTROL ALGORITHMS IN MECHATRONICS

OPTIMIZATION AND COMPUTER-AIDED TESTING OF STABILIZATION PRECISION

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The problem on maximin testing and minimax stabilization of parametrically perturbed linear control systems described up to a functional set is considered. A comparative analysis of various performance criteria of control and some methods of their realization are given. Regular and singular performance criteria of control are studied. Various estimates for results of testing are proposed. Some illustrating examples are discussed.

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1. Problem statement. Let us consider the following equations in deviations from the programmed motion:

$$
\begin{aligned}
&\frac{dx}{dt} = \left(A_0 + \sum_{i=1}^{m} A_i p_i\right) x + \left(B_0 + \sum_{i=1}^{m} B_i p_i\right) u, \quad x(t_0) \in X_0 \\
&u(\cdot) \in U = \left\{ u(\cdot) \in \tilde{B} \mid \left|u_j(t)\right| \leqslant \mu_0 \right\} \\
&p(\cdot) \in P = \left\{ p(\cdot) \in B \mid \left|p_i(t)\right| \leqslant \nu_0, \quad \left|\dot{p}_i(t)\right| \leqslant \nu_1 \right\}
\end{aligned}
\tag{1}
$$

Here $x(t)$ is the n-vector of deviations from a given trajectory; $p(\cdot)$ is the m-dimensional vector function of parametric perturbations; $u(\cdot)$ is the s-dimensional vector function of stabilizing controls; $x(t_0) \in X_0$ is the set of initial deviations; μ_0, ν_0, and ν_1 are the given parameters describing the resources of control and the parametric perturbations. Both programmed trajectories (if there exist several of them) and possible errors in some parameters of the system (or variations of these parameters during the process of operation of the system) may be considered as time-varying parametric perturbations.

It is assumed that system (1) is stabilizable and completely controllable with any constant value of $p \in P$ ($x(t) \to 0$ as $t \to \infty$) and for any $p \equiv \text{const} \in P$ if the corresponding stabilization algorithm $u(x,t) \in U$ is used.

Suppose the precision of stabilization is described by the functional

$$
J = x^T(t_k) S x(t_k) + \int_{t_0}^{t_k} \left(x^T G x + u^T N u\right) dt = x^T(t_k) S x(t_k) + \int_{t_0}^{t_k} \mathcal{G}\big(x(t), u(t)\big)\, dt \tag{2}
$$

where t_k is the instant of time when the given manifold in the space of deviations is first reached (in particular, it is possible to consider the case of a fixed t_k, $0 < t_k \leqslant \infty$); G, N, and S are constant matrices such that $G^T = G \geqslant 0$, $N^T = N \geqslant 0$, and $S^T = S \geqslant 0$.

Under the assumptions mentioned above, the following problems of synthesis and analysis of control systems are meaningful:

a) the problem of minimax stabilization

$$
\sup_{\substack{p(\cdot)\in P \\ x(t_0)\in X_0}} J\big(u(\cdot), p(\cdot), x(t_0)\big) \longrightarrow \inf_{u(\cdot)\in U} \tag{3}
$$

b) the problem of maximin testing

$$
\inf_{u(\cdot)\in U} J\big(u(\cdot), p(\cdot), x(t_0)\big) \longrightarrow \sup_{\substack{p(\cdot)\in P \\ x(t_0)\in X_0}} \tag{4}
$$

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Problem (3) was regarded in the literature in sufficient detail, whereas problem (4) was first formulated in [1] (its stochastic version was discussed in [2]). Problem (4) should be considered as the first stage of a testing procedure [2] (see the functional scheme represented in Figure 1).

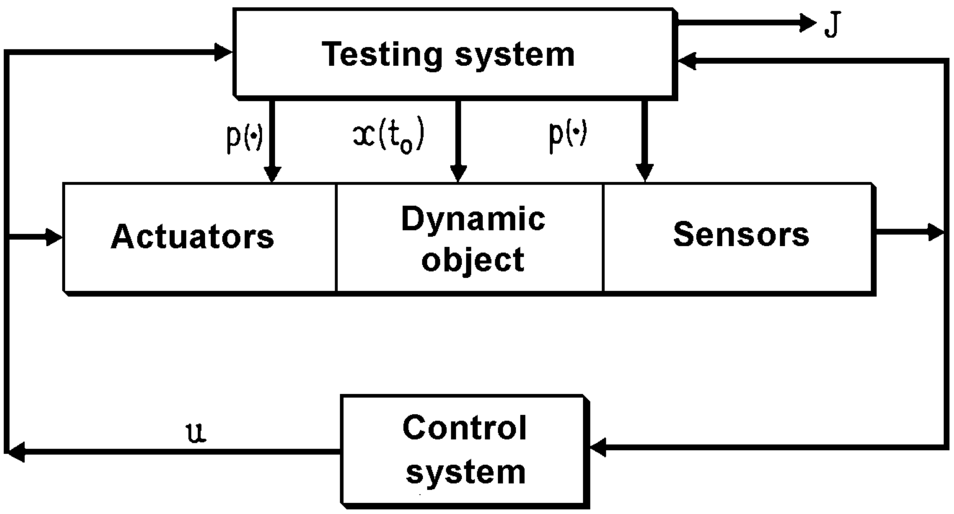

Fig. 1

The procedure of testing the quality of stabilization with the use of the automated complex consists of three stages. At the first stage, the optimal strategy $u^0(x,t)$, counterstrategy $p^0(x,u,t)$ (if they exist), and the value J_0^0 of game (3), (4) are determined (note that J_0^0 is the unimprovable qualitative lower estimate of stabilization of the programmed trajectories). The second most time-consuming stage is realized in the framework of dynamic simulation when the simulators of the medium generate a counterstrategy $x^0(t_0)$, $p^0(x,u,t)$ in response to output actions $u(x,t)$ of a stabilization algorithm being tested. At the third stage, the result of testing J_0 is compared with the unimprovable estimate J_0^0:

$$\kappa = \frac{J_0\big(p^0(x,u,t), u(x,t)\big)}{J_0^0} \tag{5}$$

Relation (5) is an estimate for the quality of stabilization. It should be noted that a knowledge or description of a stabilization algorithm is not necessary for realization of a testing procedure. The use of an output signal of the algorithm is only assumed (this situation takes place in practice). For example, it is possible to test a system of stabilization when a human is a component of this system.

Thus, when $p^0 = p^0(x,u,t)$, the first stage of testing is game (3), (4) with discrimination. Let us consider some special cases.

2. Regular problems of stabilization and testing. It is assumed that

$$\nu_0 = 0,\quad \mu_0 = \infty,\quad t_0 = 0,\quad t_k = \infty,\quad S = 0,\quad N > 0,\quad \operatorname{rank}(B_0, A_0B_0, \ldots, A_0^{n-1}B_0) = n$$

Let us consider problem (4) of maximin testing. As is known [3], the inner problem has the only solution $u^0 = -N^{-1}B^T\mathcal{L}x$ and a minimal value of the functional J_0 is equal to $x(0)^T\mathcal{L}x(0)$, where $\mathcal{L}$ is the symmetric positive definite matrix satisfying the Riccati algebraic equation

$$\mathcal{L}B_0N^{-1}B_0^T\mathcal{L} - (A_0^T\mathcal{L} + \mathcal{L}A_0) - G = 0$$

If $X_0 = \{x(0) \in R^n,\ |x(0)| \leqslant r\}$, then

$$J_0^0 = \max_{|x(0)|\leqslant r} J_0\big(x(0)\big) = r\mu_{\max} \tag{6}$$

where $\mu_{\max} = \max\limits_{1\leqslant i\leqslant n}\{\mu_i\}$ and μ_i are the roots of the equation $\det(\mu E - \mathcal{L}) = 0$. The solution of problem (6) (i.e., the eigenvector $x^0(0)$ of the matrix $\mathcal{L}$ corresponding to the eigenvalue $\mu_{\max}$) is determined from the equation $\mathcal{L}x^0(0) = \mu_{\max}x^0(0)$.

Let us use this result and reduce the original problems (3) and (4) to the finite-dimensional game

$$\max_{|x(0)|\leqslant r} J\big(K, x(0)\big) \longrightarrow \min_{K\in Q} \tag{7}$$

$$\min_{K\in Q} J\big(K, x(0)\big) \longrightarrow \max_{|x(0)|\leqslant r} \tag{8}$$

Here $u = Kx$ and

$$Q = \big\{K|\ \operatorname{Re}\lambda_l \leqslant -\alpha_0 < 0,\ \det\big(\lambda_l E - (A_0 + B_0 K)\big) = 0,\ \ l = 1, \dots, n\big\} \tag{9}$$

Since the optimal solution $K_0 = -N^{-1}B\mathcal{L}$ does not depend on $x(0)$, we have

$$\max_{|x(0)|\leqslant r}\min_{K\in Q} J\big(K, x(0)\big) = \max_{|x(0)|\leqslant r} J\big(K_0, x(0)\big) \geqslant \min_{K\in Q}\max_{|x(0)|\leqslant r} J\big(K_0, x(0)\big)$$

On the other hand, the following inequality is always fulfilled:

$$\max_{|x(0)|\leqslant r}\min_{K\in Q} J\big(K, x(0)\big) \leqslant \min_{K\in Q}\max_{|x(0)|\leqslant r} J\big(K, x(0)\big)$$

Thus, game (7), (8) has the saddle point $\big(K_0, x^0(0)\big)$:

$$\max_{|x(0)|\leqslant r}\min_{K\in Q} J\big(K, x(0)\big) = \min_{K\in Q}\max_{|x(0)|\leqslant r} J\big(K, x(0)\big) = J\big(K_0, x^0(0)\big) = J_0^0 = r\mu_{\max}$$

Example 1. Let us assume that system (1) and functional (2) are as follows:

$$\ddot{x}_1 = u_1, \quad J = \int_0^\infty \big(x_1^2 + \dot{x}_1^2 + u_1^2\big)\,dt$$

The functional J has the following kinematic meaning:

$$J = \int_0^\infty \big(x_1^2 + \dot{x}_1^2 + \ddot{x}_1^2\big)\,dt$$

The problems of stabilization and testing have the solutions

$$u_1^0 = -x_1 - \sqrt{3}\,\dot{x}_1, \quad x_1^0(0) = \pm\frac{1}{2}, \quad \dot{x}_1^0(0) = \pm\frac{1}{2} \ \text{ for } \ r = 1 \ \text{ and } \ J_0^0 = 1 + \sqrt{3}$$

The dynamic properties of the solution to the finite-dimensional game are as follows:
a) the closed system is oscillatory with the natural frequency $\omega_0 = \dfrac{1}{2}$;
b) the best degree of damping of the transient process for the functional being considered is $\alpha_0 = \dfrac{\sqrt{3}}{2}$.

3. The singular problem of testing. It is assumed that

$$\nu_0 = 0, \quad \mu_0 = \infty, \quad t_0 = 0, \quad t_k = \infty, \quad S = 0, \quad N = 0, \quad s = 1, \quad \det\big(B_0, A_0B_0, \dots, A_0^{n-1}B_0\big) \neq 0$$

Let us consider problem (4) of testing. Following [4], we shall regard measurable controls $u(t)$ $(0 \leqslant t < \infty)$ as admissible if $\big|u(\cdot)\big| \in L_2(0, \infty)$. In the case of a scalar control $u_1(t)$, A_0 and G are matrices of order n and B_0 is a vector.

It is assumed that the following frequency conditions are fulfilled (here $A_\lambda = \lambda E_n - A_0$ and $\delta(\lambda) = \det A_\lambda$):
1) $\mathcal{G}(A_{i\omega}^{-1}B_0u_1, u_1) > 0$ for any $\omega \in R^1$ $(\delta(i\omega) \neq 0)$ and for any $u_1 \neq 0$;
2) $\lim\limits_{\omega\to 0}\big|\delta(i\omega)\big|^2\mathcal{G}(A_{i\omega}^{-1}B_0u_1, u_1) > 0$ for any $u_1 \neq 0$ as $\omega \to 0$.

As is shown in [4], no solution of the inner problem in the class L_2 exists. Nevertheless, $\inf J = x^T(0)Hx(0)$. The matrix H and the vector h are uniquely determined from equations (10) and (11), respectively:

$$2\operatorname{Re} x^*H(A_0x + B_0u_1) + \mathcal{G}(x, u_1) = |h^*x|^2 \ \text{ for any } \ x,\ u_1 \tag{10}$$

$$h^*(A_0 - \lambda E_n)^{-1} B_0 = \frac{\psi(\lambda)}{\delta(\lambda)} \quad \text{for any} \quad \lambda \tag{11}$$

Here $\psi(\lambda)$ is a Hurwitz polynomial, $\varphi(i\omega) = \left|\psi(i\omega)\right|^2$, and the polynomial $\varphi(\lambda)$ is determined by the relation

$$\mathcal{G}(A_{i\omega}^{-1} B_0 u_1, u_1) = \varphi(i\omega)\left|\delta(i\omega)\right|^{-2} |u_1|^2 \tag{12}$$

If $\inf J = x^T(0) H x(0)$ is known, then it is possible to find an unimprovable lower estimate and an optimal counterstrategy $x^0(0)$ allowing us to realize the testing procedure. Note that it is natural to consider this estimate as the best result with respect to the precision of stabilization.

Example 2. Let us consider the system $\ddot{x}_1 = u_1$ from the first example but with the degenerate functional

$$J = \int\limits_0^\infty \left(x_1^2 + \dot{x}_1^2\right) dt$$

Obviously, the frequency conditions are fulfilled:

1) $\mathcal{G}(A_{i\omega}^{-1} B_0 u_1, u_1) = \dfrac{u_1^2(\omega^2+1)}{\omega^4} > 0$ for any $\omega \in R^1$ $(\delta(i\omega) \neq 0)$ and for any $u_1 \neq 0$;

2) $\lim\limits_{\omega \to 0} \left|\delta(i\omega)\right|^2 \mathcal{G}(A_{i\omega}^{-1} B_0 u_1, u_1) = u_1^2(\omega^2+1) > 0$ for any $u_1 \neq 0$ as $\omega \to 0$.

The polynomial $\varphi(\lambda) = \lambda^2 + 1$ is determined from relation (12). Since $\varphi(i\omega) > 0$, there exists the only Hurwitz polynomial $\psi = 1$ satisfying the factorization condition $\varphi(i\omega) = \left|\psi(i\omega)\right|^2$ and the following normalization condition: the positiveness of its leading coefficient.

The vector $h^T = (-1, -1)$ is determined from relation (11). The matrix H is found from equation (10) with $u = c^* x$, where c is a vector such that the matrix $A_c = A_0 + B_0 c^*$ does not have eigenvalues arranged symmetrically with respect to the imaginary axis. It follows from controllability that the matrix A_c may have any prescribed spectrum if the vector c is chosen appropriately. Thus, the matrix H takes the form

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, $\inf\limits_{u(\cdot)\in U} J = x(0)^* H x(0) = x_1^2(0) + \dot{x}_1^2(0)$. The unimprovable estimate $\max\limits_{|x(0)|\leqslant 1} \inf\limits_{u(\cdot)\in U} J = 1$ is reached with $x_1(0) = \pm\dfrac{1}{\sqrt{2}}$ and $\dot{x}_1(0) = \pm\dfrac{1}{\sqrt{2}}$.

4. The stabilization problem with $N = 0$. The stabilization problem is considered with regard to the result given in [4] and taken in the reduced form (7) with $u = Kx$. The assumptions of Section 3 of this paper are valid. Now we continue the reduction to the problem of nonlinear programming for the case of the diagonal matrix $G = G_s = \operatorname{diag}\{g_i\}$, where $g_i > 0$, $i = 1, \ldots, s$, $g_j = 0$, $j = s+1, \ldots, n$.

Let us consider the differential matrix equation

$$\dot{Z} = \tilde{A}^T Z + Z\tilde{A}, \quad Z(0) = G_s \tag{13}$$

where $\tilde{A} = A_0 + B_0 K$; its solution is $Z = e^{\tilde{A}^T t} G_s e^{\tilde{A} t}$.

For any $K \in Q$, the integral $\int\limits_0^\infty Z(t)\,dt$ is convergent, since $e^{\tilde{A}t} \to 0$ (the real parts of all eigenvalues of the matrix $\tilde{A} = A_0 + B_0 K^T$ are negative by virtue of condition (9)). Integrating equation (13), we obtain

$$A^T \left(\int\limits_0^\infty Z(t)\,dt\right) + \left(\int\limits_0^\infty Z(t)\,dt\right) A = \int\limits_0^\infty \dot{Z}(t)\,dt = Z(\infty) - Z(0) = -G_s, \quad Z(\infty) = 0$$

Thus, $\tilde{H} = \int\limits_0^\infty Z(t)\,dt$ is a solution of the Lyapunov matrix equation

$$\tilde{A}^T(K)\tilde{H} + \tilde{H}\tilde{A}(K) = -G_s \tag{14}$$

By virtue of asymptotic stability of the matrix $\tilde{A}$, the symmetric matrix $\tilde{H}$ is the only solution of equation (14) [5]. The quadratic form

$$x^T(0)\tilde{H}x(0) = \int_0^\infty x^T(0)e^{\tilde{A}^T t}G_s e^{\tilde{A}t}x(0)\,dt = \int_0^\infty \sum_{i=1}^s x_i^2(t)\,dt$$

is positive definite. Since $x(0) \neq 0$, the matrix $\tilde{H}(K)$ is positive definite for any $K \in Q$. The functional J is a quadratic form of initial conditions: $J = x^T(0)\tilde{H}(K)x(0)$. For any symmetric positive definite matrix, the inequality $x^T(0)\tilde{H}x(0) \leqslant \mu_{\max}(K)r^2$ takes place, where $\mu_{\max}$ is the maximal root of the equation $\det(\mu E - H(K)) = 0$.

Thus, we constructed the reduction to the problem of nonlinear programming:

$$\min_{K\in Q} J(K) \Longrightarrow \min_{K\in Q} r\mu_{\max}(K)$$

In order to solve this problem numerically, various methods are used (for example, the method of quadratic approximations [6]).

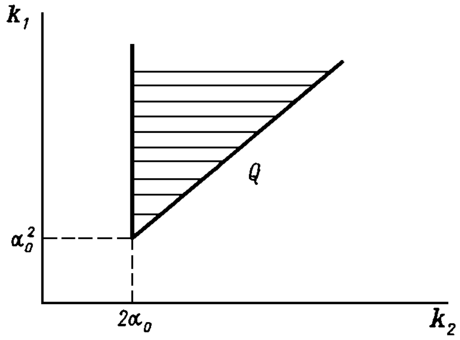

Fig. 2

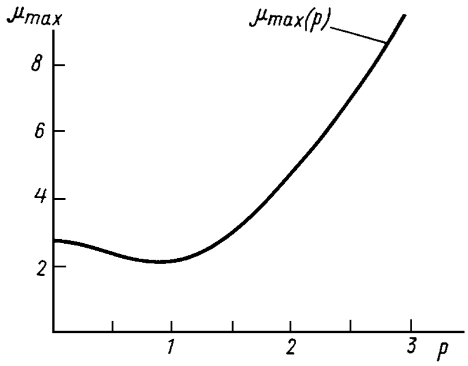

Fig. 3

Example 3. Let us consider the system and functional from Example 2 for the case when the set Q (Figure 2) is as follows:

$$Q = \{(k_2, k_1)^T \in R^2 \mid k_2 \geqslant 2\alpha_0,\ k_1 \geqslant \alpha_0 k_2 - \alpha_0^2\}$$

The matrix $\tilde{H}$ being a solution to equation (14) has the form

$$\tilde{H} = \begin{pmatrix} \dfrac{k_2^2 + k_1^2 + k_1}{2k_1k_2} & \dfrac{1}{2k_1} \\ \dfrac{1}{2k_1} & \dfrac{k_1+1}{2k_1k_2} \end{pmatrix}$$

The eigenvalues of the matrix $\tilde{H}$ are

$$\mu_{1,2} = \frac{(k_1+1)^2 + k_2^2 \pm \sqrt{2(k_2^2 - k_1^2) + (k_2^2 + k_1^2)^2 + 1}}{4k_1k_2}$$

$$\mu_{\max} = \frac{(k_1+1)^2 + k_2^2 + \sqrt{2(k_2^2 - k_1^2) + (k_2^2 + k_1^2)^2 + 1}}{4k_1k_2}$$

If $\alpha_0 > \sqrt{2+\sqrt{5}}$, then it is possible to prove analytically that the minimum is reached at a corner point (Figure 2). Our further numerical analysis allowed us to find the optimal degree of damping $\alpha_0^0 \approx 2.31$ and to construct the minimax algorithm of stabilization $u_1^0 = -5.3384\,x_1 - 4.6209\,\dot{x}_1$. It should be noted that in this case the optimal closed system is nonoscillatory. The optimal value of the functional J^0 is 1.1274. The optimal counterstrategy in the minimax problem is $\tilde{x}_1^0(0) = \pm 0.9956$, $\dot{\tilde{x}}_1^0(0) = \pm 0.0934$.

The last two examples allow us to illustrate the procedure of testing the precision of stabilization. At the first stage, the unimprovable estimate $J_0 = 1$ (or the best result of testing) and the optimal counterstrategy

$$x_1(0) = \pm\frac{1}{\sqrt{2}}, \quad \dot{x}_1(0) = \pm\frac{1}{\sqrt{2}}$$

are found. At the second stage, the testing procedure is realized.

Let us assume that the minimax algorithm $u_1^0 = -5.3384\, x_1 - 4.6203\, \dot{x}_1$ is tested and the testing strategy $x_1^0(0)$, $\dot{x}_1^0(0)$ found at the first stage is used. Then we obtain the result $J_0 = 1.1236$, which is better than the minimax result $J^0 = 1.1274$. At the third stage, $J_0 = 1.1236$ is compared with the best result $J_0^0 = 1$.

Thus, the inequalities

$$\max_{|x(0)|\in X_0} \inf_{u(\cdot)\in U} J\big(u, x(0)\big) \leqslant \inf_{u(\cdot)\in U} \max_{|x(0)|\in X_0} J\big(u, x(0)\big) \leqslant \min_{K\in Q} \max_{|x(0)|\in X_0} J\big(K, x(0)\big)$$

hold. In the absence of a saddle point, hence, a solution of the minimax problem may be used as an upper estimate of the result of testing; if this estimate is satisfactory, then the second stage of testing may be omitted.

5. Algorithms for solving the regular problems in the case of time-varying parametric perturbations. It is assumed that

$$\nu_1 = 0, \quad t_0 = 0, \quad t_k = \infty, \quad S = 0, \quad N > 0, \quad \mu_0 = \infty$$

5.1. At the first stage, problem (4) of maximin testing is reduced to solving the following three problems:

$$J\big(u(\cdot), p, x(0)\big) \xrightarrow{\text{A}} \min_{u(\cdot)\in U} \Longrightarrow J\big(u^0, p, x(0)\big) \xrightarrow{\text{B}} \max_{x(0)\in X} \Longrightarrow J\big(u^0, p, x^0(0)\big) \xrightarrow{\text{C}} \max_{p\in P}$$

A. Solving the inner problem $\min\limits_{u(\cdot)\in U} J$ with the aid of Kalman's result, we obtain the quadratic form $x(0)^T \mathcal{L}(p) x(0)$, where $\mathcal{L}(p)$ is a positive definite solution of the Riccati algebraic equation

$$\mathcal{L}(p)A(p) + A(p)^T\mathcal{L}(p) + G - \mathcal{L}(p)B(p)^T N^{-1} B(p)\mathcal{L}(p) = 0$$

Here $A = A_0 + \sum\limits_{i=1}^{m} A_i p_i$ and $B = B_0 + \sum\limits_{i=1}^{m} B_i p_i$. Since the system is controllable, such a solution is unique. The optimal control has the form $u^0 = -N^{-1}B^T\mathcal{L}(p)x$.

B. A solution to the inner problem of finding $\max\limits_{x(0)\in X}$ is the expression $\mu_{\max}(p)r$, where $\mu_{\max}(p)$ is the maximal root of the equation $\det\left(\mu E - \mathcal{L}(p)\right) = 0$.

C. Thus, the original problem is reduced to the outer one of finding $\max\limits_{p\in P} r\mu_{\max}$. The latter problem is that of nonlinear programming and may be solved by one of the standard methods. The maximal value of the functional is reached with the worst perturbation p^0.

The worst initial condition $x^0(0)$ is the eigenvector of the matrix $\mathcal{L}(p)$ corresponding to the maximal eigenvalue $\mu_{\max}(p)$ with the worst perturbation p^0. $J_0^0 = r\mu_{\max}(p^0)$ is a solution of the maximin problem and is regarded as the best result.

Example 4. Let us consider the control system $\ddot{x}_1 + px_1 = u_1$ (where $p = \omega^2 + p_1$ and $0 < p_- \leqslant p \leqslant p_+$) and the functional

$$J = \int\limits_0^\infty \left(x_1^2 + \dot{x}_1^2 + u^2\right) dt$$

In accordance with the procedure of reduction, we obtain the positive definite solution of the system of Riccati equations

$$l_{12} = -p + \sqrt{p^2+1}$$
$$l_{22} = \sqrt{-2p + 1 + 2\sqrt{p^4+1}}$$
$$l_{11} = \sqrt{p^4+1}\sqrt{-2p + 1 + 2\sqrt{p^2+1}}$$

where l_{ij} are the elements of the matrix $\mathcal{L}(p)$:

$$\mathcal{L}(p) = \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}$$

The optimal control is $u^0 = -l_{12}x_1 - l_{22}\dot{x}_1$. The inner problem has the solution

$$\max_{|x(0)|\leqslant 1} J = \mu_{\max}(p) = \frac{1}{2}\left(1+\sqrt{p^4+1}\right)\sqrt{-2p^2+1+2\sqrt{p^4+1}} + \frac{1}{2}\sqrt{-2p^6+5p^4-4p^2+2+(2p^4-4p^2+2)\sqrt{p^4+1}}$$

Thus, the original problem is reduced to that of finding a maximal value of the function $\mu_{\max}(p)$ on the segment $0 \leqslant p_- \leqslant p \leqslant p_+$. The maximal value of the functional is reached for the worst perturbation p^0.

It is interesting to note that the maximin value of the functional is reached either at the lower or at the upper bound of the perturbation parameter p, depending on a given interval of variations of parametric perturbations (Figure 3).

5.2. For the problem of minimax stabilization, we restrict ourselves to a stabilization algorithm of the type $u = Kx$. Then, the functional J takes the form

$$J = \int_0^\infty \left(x^T G x + u^T N u\right) dt = \int_0^\infty x^T \tilde{G} x \, dt, \quad \text{where} \quad \tilde{G} = G + K^T N K$$

$$\max_{|x(0)|\leqslant r} \int_0^\infty x^T \tilde{G} x \, dt = \max_{|x(0)|\leqslant r} x(0)^T \tilde{H}(p,K) x(0) = \mu_{\max}(p,K)$$

where $\mu_{\max} = \max\limits_{1\leqslant i\leqslant n}\{\mu_i\}$, μ_i are the roots of the equation $\det\left(\mu E - \tilde{H}(p,K)\right) = 0$, and $\tilde{H}$ is a solution of Lyapunov equation (14). For the nondifferentiable functional, thus, we have the finite-dimensional extremal problem

$$\max_{p\in P} r\mu_{\max}(p,K) \longrightarrow \min_{K\in Q}$$

whose numerical solution may be found using one of the procedures given in [7].

6. The case of the differential game. It is assumed that

$$t_k < \infty, \quad \mu_0 < \infty, \quad \nu_0 < \infty, \quad \nu_1 = \infty, \quad x(0) = a$$

Let us use the theorem on existence of a saddle point in the differential game with discrimination for a fixed t_k [8]:

$$\max_{p\in P}\min_{u\in U} J(u(\cdot), p(\cdot)) = \min_{u\in U}\max_{p\in P} J(u(\cdot), p(\cdot)) = J_0(u^0(x,t), p^0(u,x,t))$$

Regarding the inner problem on maximum witout the special control, we obtain the optimal counterstrategy

$$p_i^0(u,x,\psi) = \nu_0 \operatorname{sign} \psi^T (A_i x + B_i u)$$

where ψ is a solution of the corresponding dual system. Thus, in the general case of realization of the testing procedure, we have to use the output signal of the stabilization algorithm.

7. Optimal parameters and computer-aided testing of a locomotive robot. In recent years, international competitions in robotics become to grow in popularity. Each team demonstrates a locomotive robot that has to perform a preassigned series of actions, moving autonomously. In particular, one of the necessary conditions is the "ability" of the robot to move along a trajectory marked on the training ground. The time it takes for the robot to pass through all the segments of the trajectory is among the most essential parameters to judge the success of the corresponding team in the competition. In addition, the robot should be "able" to choose the correct trajectory according to rules of the competition.

Let us consider the problem of finding optimal coefficients for the control of a robot under the worst initial conditions (the problem of minimax stabilization) and the problem of computer-aided testing.

7.1. The equations of motion of a locomotive robot with two active and one passive wheels (see Figure 4, B) are as follows [9]:

$$\begin{aligned}
\dot{x} &= v\cos\alpha - h\omega\sin\alpha \\
\dot{y} &= v\sin\alpha + h\omega\cos\alpha \\
\dot{\alpha} &= \omega \\
\dot{\omega} &= \frac{mb\omega v}{J} - \frac{2a^2\omega\sigma}{r^2 J} + \frac{\chi}{rJ}\,a\,(u_R - u_L) \\
\dot{v} &= -b\omega^2 - \frac{2\sigma}{r^2 m} + \frac{\chi}{rm}\,(u_R + u_L)
\end{aligned} \qquad (15)$$

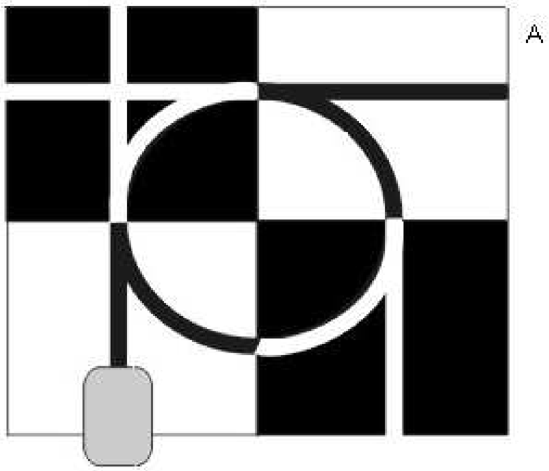

Fig. 4, A

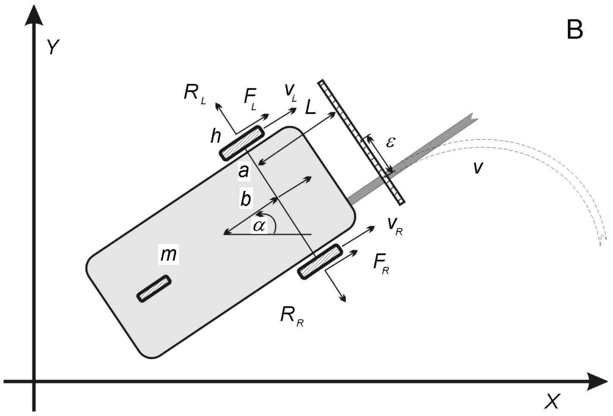

Fig. 4, B

Here x and y are the coordinates of the center of the fore wheelset axle, α is the angle characterizing the attitude of the robot, v is the linear velocity of the center of the fore wheelset axle, u_R is the control of the right wheel, and u_L is the control of the left wheel.

The parameters of the robot are given in the table.

Variable	Magnitude	Description
V_c	1.5	The velocity of the robot (m/sec)
a	0.40	The distance between the wheels (m)
b	0.05	The distance between the center of mass and the center of the fore wheelset axle (m)
h	0.10	The distance between the center of the fore wheelset axle and the point of attachment of the sensors (m)
m	4.5	The mass of the robot (kg)
r	0.08	The radius of the wheels (m)
R	0.35	The radius of inertia of the robot (m)
χ	0.01	The coefficient of viscous friction
σ	0.009	The coefficient of the counterelectromotive force

Now we consider the following three types of trajectories: 1) the first one corresponds to the case when the robot moves along the straight line parallel to the axis OX, 2) the second one corresponds to the case when the robot moves along the straight line parallel to the axis OY, and 3) the third one corresponds to the case when the robot moves along a segment of the circle of diameter 1 m centered at the vertex of one of the squares on the training ground (see Figure 4, A). The equations of programmed motions can be written down as

$$\begin{array}{llll} x = v_0 t + x_0, & y = 0, & \alpha = 0, & v = v_0 \\ \omega = 0, & x = 0, & y = v_0 t + y_0, & \alpha = \pi/2 \\ v = v_0, & \omega = 0, & x = R\cos\varphi, & y = R\sin\varphi \\ \alpha = \omega_c t + \pi/2, & v = \omega_c R, & \omega = \text{const}, & \varphi = \varphi_0 - \pi \end{array}$$

The equations in deviations

$$\dot{x} = A\,x + B\,\Delta u \tag{16}$$

can be represented for all the above cases as follows.

1) The motion parallel to the axis OX:

$$A_X = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & v_0 & 0 & h \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{2\sigma}{r^2 m} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{mbv_0}{J} - \dfrac{2a^2\sigma}{r^2 J} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{a\chi}{Jr} \end{pmatrix} \tag{17}$$

2) The motion parallel to the axis OY:

$$A_Y = \begin{pmatrix} 0 & 0 & v_0 & 0 & -h \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{2\sigma}{r^2 m} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{mbv_0}{J} - \dfrac{2a^2\sigma}{r^2 J} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{a\chi}{Jr} \end{pmatrix} \tag{18}$$

3) The motion along the circle (in the counterclockwise direction):

$$A = \begin{pmatrix} 0 & 0 & -(v_c\sin\alpha_c + h\omega_c\cos\alpha_c) & \cos\alpha_c & -h\sin\alpha_c \\ 0 & 0 & v_c\cos\alpha_c - h\omega_c\sin\alpha_c & \sin\alpha_c & h\cos\alpha_c \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{2\sigma}{r^2} & -2b\omega_c \\ 0 & 0 & 0 & \dfrac{mb\omega_c}{J} & \dfrac{mbv_c}{J} - \dfrac{2a^2\sigma}{r^2 J} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{\chi a}{Jr} & -\dfrac{\chi a}{Jr} \\ \dfrac{\chi}{mr} & \dfrac{\chi}{mr} \end{pmatrix} \tag{19}$$

It can be seen from the above that the first two systems are autonomous (they do not depend on time, in contrast to the third one).

7.2. In order to find the optimal parameters of minimax stabilization, we consider the variables (y, α, ω) for case (17) and (x, α, ω) for case (18) (see Section 2 of this paper). Then, the matrices A_X and A_Y become of order 3:

$$A_X = \begin{pmatrix} 0 & 1.5 & 0.1 \\ 0 & 0 & 1 \\ 0 & 0 & -0.3922 \end{pmatrix}, \quad A_Y = \begin{pmatrix} 0 & 1.5 & -0.1 \\ 0 & 0 & 1 \\ 0 & 0 & -0.3922 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0.8715 \end{pmatrix}$$

Both the pairs (A_X, B) and (A_Y, B) are controllable. Now we can determine the optimal parameters according to our approach proposed in Section 2. For case (17), the optimal parameters are:

$$K = N^{-1}B^T L_0 = (3.16227, 7.45063, 4.84409)$$

All real parts of the eigenvalues of the matrix $A_X - BK$ are negative. The worst initial conditions are: $x^0(t_0) = (0.4870,\, 0.8411,\, 0.2351)$.

The behavior of deviations for case (17) is represented in Figure 5.

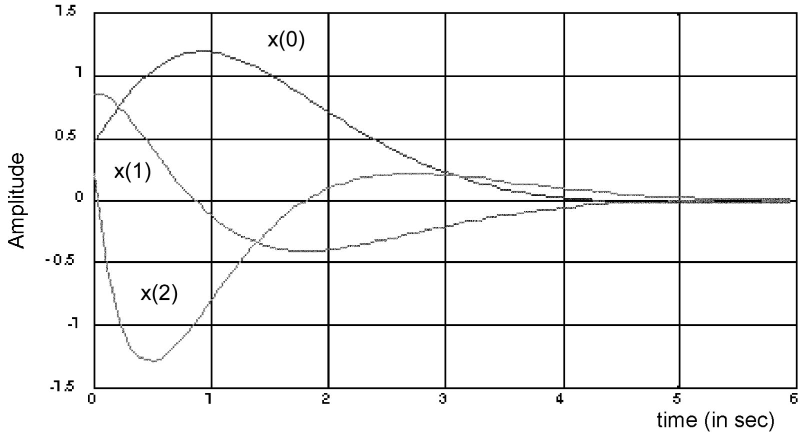

Fig. 5

7.3. In order to solve the problem under consideration, we have to apply the methods of computer-aided testing. In the context of this paper, we have to consider the problem of maximin testing

$$\inf_{u(\cdot)\in U} J\big(u(\cdot), p(\cdot), x(t_0)\big) \longrightarrow \sup_{\substack{p(\cdot)\in P \\ x(t_0)\in X_0}}$$

where $p(\cdot)$ is an m-dimensional vector function of parametric perturbations.

According to the above, the procedure of testing the quality of stabilization consists of three stages. At the first stage, we determine the optimal strategy $u^0(x,t)$, counterstrategy $p^0(x,u,t)$, and the value J_0^0 of the corresponding game (in other words, a solution of two interconnected minimax and maximin problems should be found). Recall that J_0^0 is the unimprovable lower estimate for the stabilization quality of the programmed trajectories.

At the second stage, the processes of testing are realized in the framework of dynamic simulation. As a result, a counterstrategy $x^0(t)$, $p^0(x,u,t)$ is generated in response to output actions $u(x,t)$ of the stabilization algorithm being tested.

At the third stage, the result J_0 obtained after testing is compared with the unimprovable estimate J_0^0:

$$\kappa = \frac{J_0\left(p^0(x,u,t), u(x,t)\right)}{J_0^0}$$

This result of testing is considered as an estimate for the quality of stabilization. Note that the coefficient κ should be multiplied by a factor of 10 or 100, depending on the chosen scale used for deriving estimates.

Here we consider only the first stage of testing (i.e., we determine the optimal strategy and the value of the game). Variants of the worst behavior of the moving apparatus are found with possible parameters of optimization. In this case, by the worst behavior is meant a maximally possible deviation of a robot from the programmed trajectory when the robot reaches the boundary of the square.

In order to solve the problem formulated above, let us consider system (16) for the case when $u_1 = -K^T X$ and the matrix A can be represented as (17), (18), or (19), depending on the programmed trajectory. In addition, it is assumed that the precision of stabilization is described by the functional

$$J = \sum_{p=0}^{N} \left(\int_{t_p}^{t_{p+1}} u_1^2\, dt + s_1\, \Delta x^2(t_k) + s_2\, \Delta y^2(t_k) \right)$$

Here N is the number of those squares where the trajectory is situated.

We can formulate N maximin problems for such a system. The first one consists in finding

$$\max_{|X(0)|\leqslant 1} \min_{u\in W} \left(\int_0^{t_1} u_1^2 \, dt + s_1 \, \Delta x^2(t_p) + s_2 \, \Delta y^2(t_p) \right) = J_0 \tag{20}$$

whereas the others in finding

$$\max_{1\leqslant j\leqslant R} \min_{u\in W} \left(\int_{t_p}^{t_{p+1}} u_1^2 \, dt + s_1 \, \Delta x^2(t_p) + s_2 \, \Delta y^2(t_p) \right) = J_p \tag{21}$$

where R is the number of possible "shapes" of trajectories in a given square. In our case, $R = 3$: horizontal or vertical lines and a circle are possible. Then, the solution to the general problem of finding the value J_0^0 for the first stage of testing is represented as

$$J_0^0 = J_0 + \sum_{p=1}^{N-1} J_p$$

When solving this problem, we shall consider the presence of accumulated horizontal deviations only (i.e., it is assumed that $s_1 = 1$ and $s_2 = 0$). The time t_k is regarded as a real time it takes for the apparatus to move along the corresponding part of its trajectory. Our experiments showed that the motion along a line (either horizontal or vertical) requires 1.5 sec, whereas the motion along a circle requires 1.2 sec. For problem (20), therefore, $t_1 = 1.2$ sec in the case of a rectilinear part of the trajectory and $t_1 = 1.5$ sec in the case of a circle. Problem (21) is considered on the time ranges [1.2, 2.4] and [1.5, 3.0] in the second square and, in the general case, on the time ranges $[t_p,\, t_p + 1.2]$ and $[t_p,\, t_p + 1.5]$ for straight lines and circles, respectively.

In the framework of this paper, two maximin problems were solved numerically in the first and second squares with the aid of the MAPLE 5.0 program system.

As shown in [1], problem (20) of finding $\max\limits_{|X(0)|\leqslant 1} \min\limits_{u\in W} J$ is reduced to that of finding a maximum of the quadratic form

$$\max_{|X(0)|\leqslant 1} \frac{1}{2} X^T H X = \max_{|X(0)|\leqslant 1} \frac{1}{2} X_0^T(0) H(0) X_0(0)$$

The following results were obtained for the case of the motion along the first square:

$$\max_{|X(0)|\leqslant 1} \frac{1}{2} X_0^T(0) H(0) X_0(0) = 9.7831$$

$$\begin{aligned}
x_1^0 &= x_1(0) = 0.714, & x_1(1.2) &= 5.9302\\
x_2^0 &= x_2(0) = 0.33, & x_1(1.2) &= 7.161\\
x_3^0 &= x_3(0) = 0.7, & x_1(1.2) &= 1.3914
\end{aligned}$$

For the case of the circle, the results were as follows:

$$\max_{|X(0)|\leqslant 1} \frac{1}{2} X_0^T(0) H(0) X_0(0) = 13.6476$$

$$\begin{aligned}
x_1^0 &= x_1(0) = 0.714, & x_1(1.2) &= 7.43\\
x_2^0 &= x_2(0) = 0.33, & x_1(1.2) &= 4.671\\
x_3^0 &= x_3(0) = 0.1, & x_1(1.2) &= 1.949\\
x_4^0 &= x_4(0) = 0.11, & x_1(1.2) &= 2.113\\
x_5^0 &= x_5(0) = 0.27, & x_1(1.2) &= 1.039
\end{aligned}$$

Taking into account the data obtained for the first square, we formulate problem (21) for the second square as: it is required to choose the worst trajectory (from among the three possible ones) based on the above known data, i.e., to find

$$\max_{1\leqslant j\leqslant 3} \min_{u\in W} J = \int_{1.5}^{3.0} u_1^2 \, dt + \Delta x^2(1.2)$$

for the case of a straight line and to find

$$\max_{1\leqslant j\leqslant 3}\min_{u\in W} J=\int\limits_{1.2}^{2.4} u_1^2\,dt+\Delta x^2(1.2)$$

for the case of a circle.

Thus, we obtained

$$\max_j \frac{1}{2}X_1^T(1.2)H(1.2)X_1(1.2)=11.337$$

for the rectilinear part of the trajectory and

$$\max_j \frac{1}{2}X_1^T(1.5)H(1.5)X_1(1.5)=19.2817$$

for the circle.

It can be seen from the results we obtained that an accumulation of deviations from the programmed trajectory takes place. This effect is most typical for the motion along circles.

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