STABILIZATION OF RELATIVE POSITIONS OF STEWART PLATFORMS

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The problem of finding optimal parameters for the stabilization of disturbed motions of Stewart platforms as controlled telescope segments is considered. A system of equations describing relative motions of such segments is derived. Stabilization of equilibrium positions is analyzed on the basis of equations in deviations. A numerical algorithm for optimizing the parameters of linear stabilization is proposed. This algorithm is applied to solve the problem of vertical motions with imperfect actuators. Some estimates are given for an analysis of influence of disturbed motions on the precision of stabilization of relative stationary positions. The results of numerical experiments are discussed.

1. Introduction. Different cases of Stewart platform kinematic schemes may have from three to six degrees of freedom. Since recent time, these platforms are widely used in aviation and space training devices. At present, Stewart platforms as controlled segments are applied for forming active telescope surfaces. When analyzing the possibilities of such applications, some difficulties in controlling the relative motion arise. One of these difficulties consists in the fact that during the motion of a base telescope mirror the inertial forces influence the precision of determining the relative position. Therefore, it is possible at first to choose optimal parameters of stabilization for simple situations without considering the motion and then, using these parameters, to evaluate the precision of stabilization in the presence of the inertial forces.

2. Equations of relative motion of a controlled telescope segment. The model of the telescope under consideration is represented in Figure 1. The telescope is a hemisphere of 50 meters in diameter fixed on the base. The hemisphere can change its latitude φ and longitude λ. It consists of 126 regular hexagons (segments). The mirror segments are attached to the base by three cylinders. The segments are supposed to be absolutely rigid homogeneous plane hexagons of 250 kilograms in weight each. The segment is attached to each cylinder by means of a ball-and-socket hinge. The angles between the cylinders and the base may change in time. The lengths of cylinders vary between 12 and 15 cm. Thus, each segment has three degrees of freedom. In what folows, we consider the segment situated at the center of the telescope.

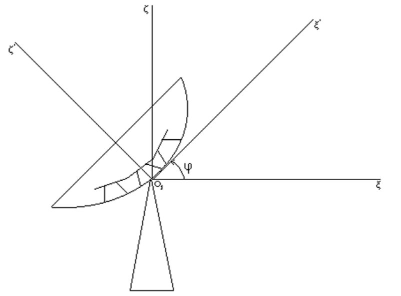

Fig. 1

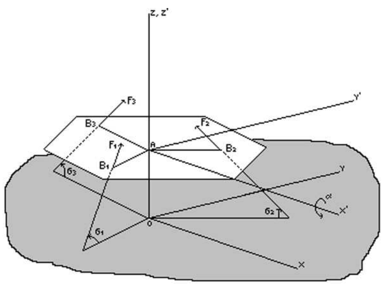

Fig. 2

Let us introduce the following coordinate systems for deriving the equations of motion of the segment (Figure 2):

1) $\xi\eta\zeta$ is the stationary coordinate system with its origin at the point O_1 (the z-axis is vertical);

2) $\xi'\eta'\zeta'$ is the coordinate system associated rigidly with the hemisphere (the axes η' and η coincide);

3) xyz is the coordinate system associated rigidly with the base (the origin of this system is the point O being the projection of the center of mass of the segment on the plane of the base and the z-axis is perpendicular to this plane);

4) $x'y'z'$ is the coordinate system with its origin at the point A (A is the center of mass of the segment, the axes x', y', z' and x, y, z are parallel);

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5) $x'''y'''z'''$ is the coordinate system associated rigidly with the segment (the point A is the origin of this system and the axes x''', y''', z''' are parallel to x, y, z in the absence of rotation of the segment).

Let us consider the case when the latitude of the hemisphere changes in time: $\varphi = \varphi(t)$ as

$$\xi\eta\zeta \xrightarrow[\eta,\eta']{\varphi} \xi'\eta'\zeta'$$

We introduce the following generalized coordinates: α is the angle of rotation of the segment about the x'-axis, β is the angle of rotation of the segment about the y''-axis, and h is the height of the center of mass of the segment above the plane of the base:

$$\xi\eta\zeta \xrightarrow[x,x'']{\alpha} x''y''z'' \xrightarrow[y'',y''']{\beta} x'''y'''z'''$$

The conventional notations for the moments of inertia are J_{xx}, J_{yy}, J_{zz} and for the mass of the segment is m.

The Lagrange method is used for deriving the equations for such a segment. The kinetic energy of the segment is

$$T = \frac{1}{2}\left(J_{xx}p^2 + J_{yy}q^2 + J_{zz}r^2\right) + \frac{1}{2}\,mv_A^2$$

where p, q, r are the projections of the absolute angular velocity on the axes x''', y''', z''', i.e.,

$$p = \dot\varphi\sin\alpha\sin\beta + \dot\alpha\cos\beta, \quad q = \dot\varphi\cos\alpha + \dot\beta, \quad r = -\dot\varphi\sin\alpha\cos\beta + \dot\alpha\sin\beta$$

The center-of-mass velocity is $\bar v_A = \dot h\bar{e_z} + [\dot\varphi\bar{e_y} \times h\bar{e_z}]$. The system of equations describing the motion of the telescope can be written down as

$$\begin{aligned}
Q_h &= m\ddot h - mh\dot\varphi^2\\
Q_\alpha &= J_1(\beta)\,\ddot\alpha + S\dot\alpha\dot\beta\sin 2\beta + \frac{1}{2}\,S\ddot\varphi\sin\alpha\sin 2\beta\\
&\quad + (S\cos 2\beta - J_{yy})\,\dot\varphi\dot\beta\sin\alpha + \frac{1}{2}\,(J_1 + J_{yy} - J_{xx} - J_{zz})\,\dot\varphi^2\sin 2\alpha\\
Q_\beta &= J_{yy}\ddot\beta + (S\cos 2\beta - J_{yy})\dot\varphi\dot\alpha\sin\alpha + J_{yy}\ddot\varphi\cos\alpha\\
&\quad - \frac{1}{2}S\dot\alpha^2\sin 2\beta - \frac{1}{2}S\dot\varphi^2\sin 2\beta\sin^2\alpha
\end{aligned} \tag{1}$$

Here the following notations are used: $J_1(\beta) = J_{xx}\cos^2\beta + J_{zz}\sin^2\beta$ and $S = J_{zz} - J_{xx}$.

Let us find the generalized forces Q_h, Q_α, Q_β. By $\bar F_1, \bar F_2, \bar F_3$ we denote the forces acting on the segment by cylinders 1, 2, 3. These forces act along the cylinders. By $\sigma_1, \sigma_2, \sigma_3$ we denote the angles between the cylinders and the plane of the base. We suppose that it is possible to control the motion of the segment by means of the forces $\bar F_1, \bar F_2, \bar F_3$:

$$\partial A = Q_h\,\partial h + Q_\alpha\,\partial\alpha + Q_\beta\,\partial\beta$$

The generalized force corresponding to the h-coordinate is

$$Q_h = \frac{\partial A}{\partial h} = F_1\sin\sigma_1 + F_2\sin\sigma_2 + F_3\sin\sigma_3 - mg\cos\varphi \tag{2}$$

The generalized force corresponding to the α-coordinate is equal to the moment M_α directed along the axis $x'||\,x''$. This moment causes the change of the angle α. The angle β changes under the influence of the moment M_β directed along the axis $y''||\,y'''$. The forces $\bar F_1, \bar F_2, \bar F_3$ create the moments $\bar M_1, \bar M_2, \bar M_3$. Therefore, $\bar M = \bar M_1 + \bar M_2 + \bar M_3$, $\bar M_\alpha = \mathrm{pr}\,(\bar M)_{x''}$, and $\bar M_\beta = \mathrm{pr}\,(\bar M)_{y''}$.

The forces $\bar F_1, \bar F_2, \bar F_3$ have the following projections in the coordinate system $Oxyz$:

$$\begin{aligned}
\bar F_1 &= \left(-\frac{1}{2}\,F_1\cos\sigma_1,\ \frac{\sqrt{3}}{2}\,F_1\cos\sigma_1,\ F_1\sin\sigma_1\right)\\
\bar F_2 &= \left(-\frac{1}{2}\,F_2\cos\sigma_2,\ -\frac{\sqrt{3}}{2}\,F_2\cos\sigma_2,\ F_2\sin\sigma_2\right)\\
\bar F_3 &= (F_3\cos\sigma_3,\ 0,\ F_3\sin\sigma_3)
\end{aligned}$$

Let us assume that $AB_1 = AB_2 = AB_3 = c$; then, the coordinates of the points B_1, B_2, B_3 in the coordinate system $Ax'''y'''z'''$ are

$$\bar{r_1} = \left(\frac{1}{2}\,c,\ -\frac{\sqrt{3}}{2}\,c,\ 0\right), \quad \bar{r_2} = \left(\frac{1}{2}\,c,\ \frac{\sqrt{3}}{2}\,c,\ 0\right), \quad \bar{r_3} = (-c,\ 0,\ 0)$$

The projections of the forces $\bar{F}_1, \bar{F}_2, \bar{F}_3$ and the coordinates of the points B_1, B_2, B_3 in the coordinate system $Ax''y''z''$ are $\bar{F_1}'', \bar{F_2}'', \bar{F_3}''$ and $\bar{r_1}'', \bar{r_2}'', \bar{r_3}''$. Thus, for $i = 1, 2, 3$ we can write down

$$\bar{M_i} = \bar{F_i}'' \times \bar{r_i}'' = (F_{iy''}r_{iz''} - F_{iz''}r_{iy''})\,\bar{e}_{x''} + (F_{iz''}r_{ix''} - F_{ix''}r_{iz''})\,\bar{e}_{y''} + (F_{ix''}r_{iy''} - F_{iy''}r_{ix''})\,\bar{e}_{z''}$$

$$M_\alpha = \mathrm{pr}\,(\bar{M})_{x''} = \sum_{i=1}^{3}(F_{iy''}r_{iz''} - F_{iz''}r_{iy''})$$

$$M_\beta = \mathrm{pr}\,(\bar{M})_{y''} = \sum_{i=1}^{3}(F_{iz''}r_{ix''} - F_{ix''}r_{iz''})$$

The generalized forces corresponding to the angles α and β are of the form

$$\begin{aligned} Q_\alpha = &-\frac{\sqrt{3}}{4}\,c\,(F_1\cos\sigma_1 - F_2\cos\sigma_2)\cos\alpha\sin\beta + \frac{\sqrt{3}}{2}\,c\,(F_1\sin\sigma_1 - F_2\sin\sigma_2)\cos\alpha \\ &-\frac{1}{2}\,c\,(F_1\sin\sigma_1 + F_2\sin\sigma_2 - 2F_3\sin\sigma_3)\sin\alpha\sin\beta - \frac{3}{4}\,c\,(F_1\cos\sigma_1 + F_2\sin\sigma_2)\sin\alpha \\ Q_\beta = &-\frac{\sqrt{3}}{4}\,c\,(F_1\cos\sigma_1 - F_2\cos\sigma_2)\sin\alpha\cos\beta - \frac{1}{4}\,c\,(F_1\cos\sigma_1 + F_2\cos\sigma_2 + 4F_3\cos\sigma_3)\sin\beta \\ &+\frac{1}{2}\,c\,(F_1\sin\sigma_1 + F_2\sin\sigma_2 - 2F_3\sin\sigma_3)\cos\alpha\cos\beta \end{aligned} \tag{3}$$

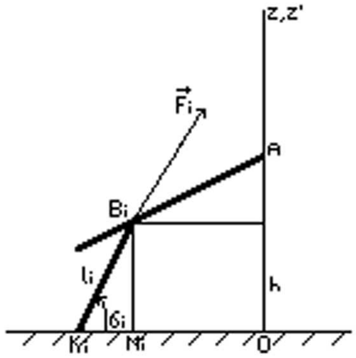

Fig. 3

Let us find $\sin\sigma_i$ and $\cos\sigma_i$, where $i = 1, 2, 3$. We will consider the vertical section of the segment along the axis Oz (Figure 3; the subscript i indicates the variables related to the ith cylinder). By K_i we denote the point of attachment of the ith cylinder to the base. Then, $d = OK_i$ and l_i is the length of the corresponding cylinder. Hence, we can write: $AB_1 = AB_2 = AB_3 = c$, $AO = h$, and $K_iO = d$. The z-coordinate of the points B_1, B_2, B_3 in the coordinate system $Ax'y'z'$ are equal to

$$\begin{aligned} r'_{1z} &= -\frac{1}{2}\,c\cos\alpha\sin\beta - \frac{\sqrt{3}}{2}\,c\sin\alpha \\ r'_{2z} &= -\frac{1}{2}\,c\cos\alpha\sin\beta + \frac{\sqrt{3}}{2}\,c\sin\alpha \\ r'_{3z} &= c\cos\alpha\sin\beta \end{aligned}$$

The z-coordinates of the points B_1, B_2, B_3 in the coordinate system $Oxyz$ are equal to: $r_{1z} = h + r'_{1z}$, $r_{2z} = h + r'_{2z}$, $r_{3z} = h + r'_{3z}$. Then, for each i we have

$$l_i^2 = r_{iz}^2 + K_iM_i^2, \quad M_iO^2 = c^2 - r_{iz}'^2$$

$$K_i M_i = K_i O - M_i O = d - \sqrt{c^2 - r'^2_{iz}}$$

$$l_i^2 = (h + r'_{iz})^2 + K_i M_i = d^2 + c^2 + h^2 + 2hr'_{iz} - 2d\sqrt{c^2 - r'^2_{iz}}$$

$$\sin\sigma_i = \frac{r_{iz}}{l_i} = \frac{h + r'_{iz}}{l_i}, \quad \cos\sigma_i = \frac{K_i M_i}{l_i} = \frac{d - \sqrt{c^2 - r'^2_{iz}}}{l_i}$$

3. Equations in deviations. Stabilization of equilibrium positions. Let us consider the case when the angle φ is constant ($\dot\varphi = 0$). Then, system (1) takes the form

$$\begin{aligned} &m\ddot h = Q_h \\ &\left(J_{xx}\cos^2\beta + J_{zz}\sin^2\beta\right)\ddot\alpha + (J_{zz} - J_{xx})\,\dot\alpha\dot\beta = Q_\alpha \\ &J_{yy}\ddot\beta - \frac{1}{2}(J_{zz} - J_{xx})\,\dot\alpha^2\sin 2\beta = Q_\beta \end{aligned} \tag{4}$$

where

$$Q_h = \sum_{i=1}^{3} F_i \sin\gamma_i - mg\cos\varphi \tag{5}$$

and

$$\begin{aligned} Q_\alpha = &-\frac{\sqrt3}{2}c\sin\beta\cos\alpha\,(F_1\cos\gamma_1 - F_2\cos\gamma_2) + \frac{\sqrt3}{2}c\cos\alpha\,(F_1\sin\gamma_1 - F_2\sin\gamma_2) \\ &-\frac{1}{2}c\sin\alpha\sin\beta\,(F_1\sin\gamma_1 + F_2\sin\gamma_2 - 2F_3\sin\gamma_3) - \frac{3}{4}c\sin\alpha\,(F_1\cos\gamma_1 + F_2\cos\gamma_2) \\ Q_\beta = &-\frac{\sqrt3}{4}c\sin\alpha\cos\beta\,(F_1\cos\gamma_1 - F_2\cos\gamma_2) - \frac{1}{4}c\sin\beta\,(F_1\cos\gamma_1 + F_2\cos\gamma_2 + 4F_3\cos\gamma_3) \\ &+\frac{1}{2}c\cos\alpha\cos\beta\,(F_1\sin\gamma_1 + F_2\sin\gamma_2 - 2F_3\sin\gamma_3) \end{aligned} \tag{6}$$

Let us consider the stationary position of the system: $\alpha = \beta = 0$ and $h = h_0$. Then,

$$\left.\begin{aligned} &Q_h|_{\alpha=\beta=0} = 0 = \sum_{i=1}^{3} F_i\sin\gamma_0 - mg\cos\varphi \\ &Q_\alpha|_{\alpha=\beta=0} = 0 = \frac{\sqrt3}{2}c\sin\gamma_0(F_1 - F_2) \\ &Q_\beta|_{\alpha=\beta=0} = 0 = \frac{1}{2}c\sin\gamma_0(F_1 + F_2 - 2F_3) \end{aligned}\right\} \Rightarrow \begin{cases} F_1 = F_2 = F_3 \\ F_1 = \dfrac{mg\cos\varphi}{3h_0}\,l_0 = F_{\text{pr}} \end{cases} \tag{7}$$

where $\sin\gamma_i(0,0,h_0) = h_0/l_0 = \sin\gamma_0$.

Let the height h be given ($h = h_0$, $\alpha = 0$, $\beta = 0$). Then, under the action of the forces $F_1 = F_2 = F_3 = D = \text{const}$, the control

$$u_i^0 = \frac{s\cdot mg\cdot\cos\varphi}{3l_0\cdot\tilde k_i}h_0$$

ensures that the system is at relative equilibrium.

Now we consider the control as $u_i = u_i^0 + \Delta u_i$ and introduce small deviations in h and α from the equilibrium position:

$$\begin{aligned} &\Delta h = h - h_0 \\ &\Delta\alpha = \alpha - \alpha_0 \\ &\Delta\beta = \beta - 0 = \beta \end{aligned} \tag{8}$$

Let us reduce the above system to the form

$$\dot X = AX + B\Delta U \tag{9}$$

where $\Delta U^T = \{\Delta u_1, \Delta u_2, \Delta u_3\}$. To accomplish this, we linearize system (4). Note that in our case the z-coordinates of the points B_i^1 are equal to zero and

$$
\begin{aligned}
&l_i^2 = l_0^2 + 2h_0 \cdot (\Delta h + k_{iz}^1), \quad l_i = l_0 + \frac{h_0}{l_0} \cdot (\Delta h + k_{iz}^1) \\
&\sin \gamma_i = \frac{h_0}{l_0} + \frac{\cos^2 \gamma_0}{l_0} \cdot (\Delta h + k_{iz}^1), \quad \cos \gamma_i = \frac{d-c}{l_0} - \frac{(d-c)h_0}{l_0^3} \cdot (\Delta h + k_{iz}^1) \\
&k_{1z}^1 = -\frac{\sqrt{3}}{2}\, c\, \Delta\alpha, \quad k_{2z}^1 = \frac{\sqrt{3}}{2}\, c\, \Delta\alpha, \quad k_{3z}^1 = 0
\end{aligned}
\tag{10}
$$

Substituting these values in (4) and (7), we obtain (9).

Let us consider the case when the rotation by the angle β is absent (i.e., $\beta = 0$). Then, the forces F_1, F_2, F_3 regarded as a linear feedback in deviations from programmed motions [4] with $0 < \tau = \text{const} \ll 1$ can be written as

$$
\begin{aligned}
&F_2 = F_1 = F_1^{\text{pr}} + c_1 \frac{h_0}{l_0} \left(\Delta h - \frac{\sqrt{3}}{2}\, \Delta\alpha \right) + c_2 \frac{h_0}{l_0} \left(\Delta \dot{h} - \frac{\sqrt{3}}{2}\, \Delta\dot{\alpha} \right), \quad u = u_i^0 + \Delta u \\
&F_3 = F_3^{\text{pr}} + c_3 \frac{h_0}{l_0}\, \Delta h + c_4 \frac{h_0}{l_0}\, \Delta \dot{h}
\end{aligned}
\tag{11}
$$

The expressions for Q_h and Q_α have the form

$$
\begin{aligned}
Q_h &= -mg \cos\varphi + \left(\frac{h_0}{l_0} + \frac{\cos^2 \gamma_0}{l_0} \Delta h \right) \\
&\times \left(\frac{mgl_0}{h_0} \cos\varphi + c_1 \left(\Delta h - \frac{\sqrt{3}}{2}\, c\, \Delta\alpha \right) + c_2 \left(\Delta \dot{h} - \frac{\sqrt{3}}{2}\, c\, \Delta\dot{\alpha} \right) + 2c_3 \Delta h + 2c_4 \Delta \dot{h} \right) \\
Q_\alpha &= -\frac{2}{2l_0}\, c\, \Delta\alpha \left(c \cos^2 \gamma_0 + \frac{d-c}{l_0} - \frac{d-c}{l_0^2} h_0 \Delta h \right) \\
&\times \left(\frac{mg \cos\varphi}{3h_0} l_0 + c_1 \frac{h_0}{l_0} \left(\Delta h - \frac{\sqrt{3}}{2}\, c\, \Delta\alpha \right) + c_2 \frac{h_0}{l_0} \left(\Delta \dot{h} - \frac{\sqrt{3}}{2}\, c\, \Delta\dot{\alpha} \right) \right)
\end{aligned}
\tag{12}
$$

The system closed by the control as a linear feedback takes the form

$$
\dot{X} = A\,(C)\, X, \quad \text{where} \quad C^T = \{c_1,\, c_2,\, c_3,\, c_4\}
\tag{13}
$$

It is required to find the optimal values of coefficients of stabilization that enter into system (13). We call coefficients optimal if they satisfy conditions of optimality and conditions for minimum of the total error under the worst perturbation.

4. Mathematical statement of the problem on optimizing the parameters of the linear stabilization algorithm. A reduction to a problem of nonlinear programming. Let us consider the problem of finding optimal (in the sense defined in Section 3 of this paper) parameters of stabilization from the system

$$
\dot{X} = A(\varphi)X + BU
\tag{14}
$$

Here X and U are $(n \times 1)$- and $(m \times 1)$-vectors, respectively $(m \leqslant n)$; A and B are $(n \times n)$- and $(n \times m)$-matrices, respectively. The initial conditions belong to a ball of radius ρ (i.e., $X^T(0)X(0) \leqslant \rho$). The control is chosen as a linear feedback: $U = -CX$. In this case, the original system can be written down in the form

$$
\dot{X} = A_1(C)X
\tag{15}
$$

Let us require that the matrix $C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}$ belong to a set Q such that:

$$
Q = \{ C \in R^m \mid \operatorname{Re}(\lambda_i) \leqslant -k_0, \quad k > 0 \} \neq \emptyset\,^3
\tag{16}
$$

[3] It is supposed that the pair $\{A, B\}$ is at least stabilizable; therefore, the set Q is not empty.

where λ_i are the eigenvalues of the matrix $A_1(C)$.

The problem of optimizing the parameters of such a system can be formulated in the following way: it is required to minimize (in $C_{ij} \in Q$, and hence, in k_0) the maximum (in initial conditions) of a functional $J(X(0), C)$, i.e., to find

$$\min_{C\in Q} \max_{|X(0)|\leqslant\rho} J$$

Assuming $C \in Q$ arbitrary, we consider the functional

$$\varphi_s(C) = \max_{X^T(0)X(0)\leqslant\rho} \int_0^\infty X^T(t)\, E_n^s X(0)\, dt \tag{17}$$

where E_n^s is a diagonal matrix of order n whose first s diagonal elements are equal to 1 and the other elements are equal to 0 [5].

Let us consider the differential-matrix equation

$$\dot{Z} = A^T Z + ZA, \quad Z(0) = E_n^s \tag{18}$$

whose solution is

$$Z = e^{A^t t} E_n^s e^{At}$$

The integral $H = \int_0^\infty Z(t)\, dt$ converges for all $C \in Q$. Indeed, $e^{At} \to 0$ as $t \to \infty$, since all eigenvalues of the matrix A have negative real parts by virtue of condition (16). Hence, equation (18) is integrable:

$$\int_0^\infty \dot{Z}(t)\, dt = A^T \left(\int_0^\infty \dot{Z}(t)\, dt\right) + \left(\int_0^\infty \dot{Z}(t)\, dt\right) A = Z(\infty) - Z(0) = -E_n^s, \quad Z(\infty) = 0$$

Thus, the quadratic form

$$X^T(0) H X(0) = \int_0^\infty X^T(0) \cdot e^{A^T t} E_n^s e^{At} X(0)\, dt = \int_0^\infty \sum_{i=1}^s x_i^2(t)\, dt$$

is positive definite and $X(0) \neq 0$; hence, the matrix $H(C)$ is also positive definite for all $C \in Q$.

The functional J is a quadratic form with respect to the initial conditions, i.e.,

$$J = X^T(0) H X(0) \tag{19}$$

The following inequality holds for any symmetric positive definite matrix [6]:

$$X^T H X \leqslant \mu_{\max}(C)\, \rho^2$$

Here μ is determined by the equation

$$\det\bigl(\mu E - H(C)\bigr) = 0 \tag{20}$$

A condition for finding $\min\limits_{C\in Q} \varphi_s(C) = \rho\mu_{\max} \to \inf\limits_{C\in Q}$ can be written down as [5]

$$\varphi_s(C) \to \min_{C\in Q}$$

Thus, the problem of finding a minimum of $\varphi_s(C)$ is reduced to that of finding a minimum of $\rho\mu_{\max}(C)$.

5. Numerical solution of the problem of vertical motion with imperfect actuators.

5.1. A numerical algorithm. Given a function $f(\bar{x}) = f(x_1, \ldots, x_n)$ of many variables[4], it is required to find $\min f(x)$ under the following constraints imposed on the arguments $x \in R^n$ [4]:

$$\begin{aligned} g_j &= 0, \quad j = 1, \ldots, m_k \\ g_j &\geqslant 0, \quad j = m_{k+1}, \ldots, m \end{aligned} \tag{21}$$

[4] In what follows, the notation $f(\bar{x}) = f(x)$ is used.

One of the methods for solving this problem is the penalty function one. According to this method, we introduce the interior penalty function

$$P(x, r_k) = f(x) + r_k \sum_{j=1}^{m_k} g_j(x)^2 + r_k \sum_{m_{k+1}}^{m} (\min(0, g_j(x))^2 \qquad (22)$$

and the exterior penalty function

$$P(x, r_k) = f(x) + r_k \sum_{j=1}^{m_k} g_j(x)^2 - \frac{1}{r_k} \sum_{m_{k+1}}^{m} \ln(g_j(x)) \qquad (23)$$

Here $r_k \to \infty$ as $k \to \infty$.

The combination of these two functions yields the mixed exterior-interior penalty function

$$P(x, r_k) = f(x) + r_k \sum_{j=1}^{m_k} g_j(x)^2 + r_k \sum_{j \in J_1} (\min(0, g_j(x))^2 - \frac{1}{r_k} \sum_{j \in J_2} \ln(g_j(x))$$

where $J_1 = \{j :\ g_j(x_0) \leqslant 0,\ m_k < j \leqslant m\}$ and $J_2 = \{j :\ g_j(x_0) > 0,\ m_k < j \leqslant m\}$.

Based on (22), the algorithm for solving problem (21) can be written as

$$P(x, r_k) = f(x) + r_k g^T(x) g(x) = f(x) + r_k \sum_{j \in J_x} g_j(x)^2 \qquad (24)$$

where J_x is the set of active constraints, $g(x)$ is the vector of all active constraints, and r_k is a sequence of increasing penalty parameters. Then,

$$\nabla_x P(x, r_k) = \nabla_x f(x) + 2 r_k J_x g(x)^T g(x) \qquad (25)$$

where $J_x g(x)$ is the Jacobian and

$$\nabla_x g(x) = \left(\frac{\partial}{\partial x_i} g(x) \right)^T$$

Expanding (25) in a Taylor series at the point $(x + p, r_k)$ (where p is an increment) and ignoring the nonlinear terms, we obtain a quadratic programming problem for the Murrey–Briggs method, where the objective function is of the form

$$\frac{1}{2} p^T B p + p^T \nabla_x f(x)$$

Here B is an appropriate approximation of the Hessian for the function $f(x)$ obtained with the quasi-Newtonian algorithm [3]. Problem (24) is reduced to finding a minimum of the quadratic form

$$\min \frac{1}{2} p^T H^{-1} p + \nabla_x f(x) p, \quad D_x g(x) p = -\frac{1}{2 r_k} \bar{w} - g(x)$$

where the vector $\bar{w}$ specifies the Lagrangian multipliers for the solution of original problem (21). Based on the above, we obtain

$$p = H (D_x g(x)^T \bar{w} - \nabla_x f(x)$$

The Wilson–Han–Powell method for solving the original problem (21) consists in finding an optimal direction d for the following subproblem:

$$\begin{aligned} &\min \frac{1}{2} d^T B d + \nabla_x f(x)\, d \\ &g(x) + \nabla_x g_j(x)^T d = 0, \quad j = 1, \ldots, m_k \\ &g(x) + \nabla_x g_j(x)^T d \geqslant 0, \quad j = m_{k+1}, \ldots, m \end{aligned} \qquad (26)$$

Here $B = \nabla_x^2 L(x, u)$ and $L(x, u) = f(x) - \sum_{j=1}^{m} u_j g_j(x)$ is the Lagrangian function.

The next iteration is defined as $x_n = x_{n-1} + \alpha d$, where the optimal direction d is found from (26) and the optimal step length α can be obtained from (24).

5.2. Numerical solution for n=3. In the absence of deviations in the angles α and β and in the absence of a delay τ, the system of equations given in Section 3 of this paper is reduced to a system of second order. If a deviation in one of the above angles takes place (for example, in the angle β), then we obtain a system of fourth order.

Let us consider the problem with delay. If $\tau \neq 0$, then it follows from (11) that

$$3\,\frac{h_0^2}{ml_0^2}\,(c_2\Delta\dot{h} + \Delta h c_1) = \Delta\ddot{h} - g\cos\varphi\,\frac{\cos^2\gamma_0}{h_0}\,\Delta h$$

Since

$$\tau\Delta\dot{p} + \Delta p = -c_1\Delta h - c_2\Delta\dot{h}$$

we have

$$\Delta\,\dddot{h} + \frac{1}{\tau}\,\Delta\ddot{h} + \left(3c_2\,\frac{h_0^2}{\tau ml_0^2} - \frac{g\cos\varphi_0\cos^2\gamma_0}{h_0}\right)\Delta\dot{h} + \left(3c_1\,\frac{h_0^2}{\tau ml_0^2} - \frac{g\cos\varphi_0\cos^2\gamma_0}{\tau h_0}\right)\Delta h = 0 \tag{27}$$

Similarly to the case $n = 2$, equation (27) can be rewritten in the form

$$\Delta\,\dddot{h} + p\Delta\ddot{h} + q\Delta\dot{h} + r\Delta h = 0$$

where

$$\begin{aligned} p &= 1/\tau \\ q &= 3c_2\,\frac{h_0^2}{\tau ml_0^2} - \frac{g\cos\varphi_0\cos^2\gamma_0}{h_0} \\ r &= 3c_1\,\frac{h_0^2}{\tau ml_0^2} - \frac{g\cos\varphi_0\cos^2\gamma_0}{\tau h_0} \end{aligned}$$

or in the form of the system $\dot{X} = A(C)X$, where

$$X^T = \{\Delta h, \Delta\dot{h}, \Delta\ddot{h}\}, \quad A(C) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -q & -p \end{pmatrix}$$

The characteristic equation of the matrix $A(K)$ is $\lambda^3 + p\lambda^2 + q\lambda + r = 0$.

Using the change of variables $\lambda = \hat{\lambda} - k_0$, we find the set Q:

$$Q = \{c_1, c_2 \mid \mathrm{Re}\,(\hat{\lambda}_i \leqslant 0),\ k_0 > 0\}.$$

Hence, the characteristic equation takes the from

$$\hat{\lambda}^3 + a_1\hat{\lambda}^2 + a_2\hat{\lambda} + a_3 = 0 \tag{28}$$

where

$$a_1 = p - 3k_0, \quad a_2 = q - 2k_0p + 3k_0^2, \quad a_3 = r - k_0q + k_0^2p - k_0^3$$

In order for real parts of the roots of characteristic equation (28) to be negative, the Hurwitz criterion should be valid:

$$\begin{cases} a_1 \geqslant 0 \\ a_1a_2 - a_3 \geqslant 0 \Leftrightarrow a_1 \geqslant 0,\ a_2 \geqslant 0,\ a_3 \geqslant 0,\ a_1a_2 - a_3 \geqslant 0 \\ a_1a_2a_3 - a_3^2 \geqslant 0 \end{cases}$$

$$\begin{cases} p - 3k_0 \geqslant 0 \\ q - 2k_0p + 3k_0^2 \geqslant 0 \\ r - k_0q + k_0^2p - k_0^3 \geqslant 0 \\ (p - 3k_0)(q - 2k_0p + 3k_0^2) - (r - k_0q + k_0^2p - k_0^3) \geqslant 0 \end{cases}$$

Hence, $Q = \{K \in R^3 \mid p - 3k_0 \geqslant 0;\ q - 2k_0p + 3k_0^2 \geqslant 0;\ r - k_0q + k_0^2p - k_0^3 \geqslant 0;\ (p - 3k_0)(q - 2k_0p + 3k_0^2) - (r - k_0q + k_0^2p - k_0^3) \geqslant 0\}$.

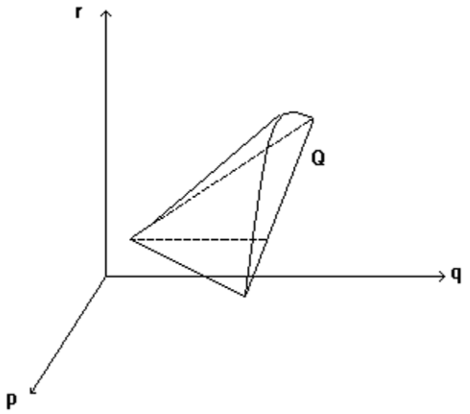

Fig. 4

If $n = 3$ and $s = 3$, then the matrix H takes the form

$$H = \begin{pmatrix} \dfrac{r^2(pq + r - 2pq^2) + q(pq^2 - qr + rp^2)}{2rpq^2} & \dfrac{qp^2 + 2r^2}{2r(pq - r)} & \dfrac{1}{2r} \\ \dfrac{qp^2 + 2r^2}{2r(pq - r)} & \dfrac{qp^2(r + p) + r(2q^2 + q + pr)}{2qr(pq - r)} & \dfrac{qp^2 + r^2 + pqr}{2rq(pq - r)} \\ \dfrac{1}{2r} & \dfrac{qp^2 + r^2 + pqr}{2rq(pq - r)} & \dfrac{p + 2r}{2r(pq - r)} \end{pmatrix}$$

Let us consider the characteristic equation $\mu^3 - a\mu^2 - b\mu - c = 0$, where

$$a = h_{11} + h_{22} + h_{33}$$
$$b = -(h_{11}h_{33} + h_{22}h_{11} + h_{33}h_{22}) + h_{13}^2 + h_{23}^2 + h_{12}^2$$
$$c = -(h_{22}h_{13}^2 + h_{11}h_{23}^2 + h_{33}h_{12}^2 + h_{33}h_{12}^2) + h_{11}h_{22}h_{33} + 2h_{13}h_{23}h_{12}$$

Its roots can be obtained by the Cardano formula:

$$\begin{cases} \mu_1 = -\dfrac{5}{3}\,a \\ \mu_2 = \dfrac{4a + 3b\sqrt{3}}{3} \\ \mu_3 = \dfrac{4a - 3b\sqrt{3}}{3} \end{cases} \Rightarrow \mu_{\max} = \mu_3$$

Hence,

$$\mu_{\max}(p, q, r) = \frac{4\,(h_{11} + h_{22} + h_{33}) - 3\sqrt{3}\,[h_{12}^2 + h_{13}^2 + h_{23}^2 - h_{33}h_{11} - h_{11}h_{22} - h_{22}h_{33}]}{3}$$

Thus, if we know an analytic expression for $\mu_{\max}$, then we can find min $\mu_{\max}$ numerically. The above problem was solved with the use the MATLAB 4.0 software (the function "constr"). The constraints were taken into account by introducing penalty terms into the objective function.

Some values of $\mu_{\max}(p_{\text{end}}, q_{\text{end}}, r_{\text{end}})$ are given below for the point $A(p_{\text{end}}, q_{\text{end}}, r_{\text{end}})$, where $\min \mu_{\max}$ is reached. The results were obtained for the following values of the geometrical parameters: $m = 100$ kg, $h_0 = 0.2$ m, $l_0 = 0.3$ m, and $g = 9.81$ m/s^2.

$k_0 = 3.0$, $\tau = 0.1$

p_0	q_0	r_0	p_{end}	q_{end}	r_{end}	$\mu_{\max}$
10	33	36	9	27	27	4.2346
100	240	500	8.9999	26.999	26.999	$\tau = 0.01$
1000	6000	9500	9.0000	27.000	27.000	$\tau = 0.001$

The values of stabilization coefficients c_1 and c_2 in the case $\tau = 0.1$

φ_0 (grad)	c_1	c_2
0	120.0833	1497.5
30	108.6616	1314.925
45	94.2192	1098.29
60	75.4167	816.25

$k_0 = 6.0$, $\tau = 0.2$

p_0	q_0	r_0	p_{end}	q_{end}	r_{end}	$\mu_{\max}$
20	240	500	18	108	216	18.13199
100	2400	5000	18.000	108.00	216.00	$\tau = 0.01$
1000	23500	49500	17.(9)	107.(9)	215.(9)	$\tau = 0.001$

It is easy to see from (23) that e^{At} tends to zero more rapidly when k_0 increases ($k_0 > 0$). This allows us to assume that the stability of the system is reached faster if larger values of k_0 are chosen. However, this is a wrong conclusion, since the influence of perturbations on the system increase with increasing k_0. This assertion is confirmed by the data given in the above tables: the value of μ_{max} is smaller for $k_0 = 3$ than that obtained for $k_0 = 6$. Hence, it is better to choose $k_0 = 3$, trying to reach the stability in a larger time, but under smaller perturbations.

6. Estimation of the influence of disturbed motions on the precision of stabilization of relative stationary positions. Let us formulate the case under consideration in the context of Alexandrov's theorem on absolute stability given in the book "Mathematical Problems of Dynamic Simulation of Aerospace Flights" [3].

Let us consider the system

$$\dot{x} = A_c(p, v, w)\, x, \quad \dot{p} = v, \quad \dot{v} = w \tag{29}$$

where $p = p(t)$ is a parametric perturbation. Suppose that $A_c(p, 0, 0)$ is a Hurwitz matrix for all $p \in [p_-, p_+]$. Let $H^T(p) = H(p) > 0$ for all $p \in [p_-, p_+]$ and

$$L = \frac{\partial H}{\partial p}\, \dot{p} + A_c^T H + H A_c$$

In order to find $H(p)$, we shall use the Lyapunov equation

$$A_c^T(p, 0, 0)H + HA_c(p, 0, 0) = -E$$

Now we introduce the set

$$V = \{p_- \leqslant p \leqslant p_+,\ v_-(p) \leqslant v \leqslant v_+(p),\ 0 \in [v_-, v_+],\ -w_+ \leqslant w \leqslant w_+\}$$

where $p(t)$ is a monotone or piecewise monotone function, $v_\pm(p)$ are piecewise holomorphic continuous functions on $[p_-, p_+]$, $v(t)$ may change its sign for a finite or countable number of times on $(0, \infty)$, and $w(t)$ is a piecewise holomorphic function.

Definition. If the solution of system (29) is asymptotically stable for all $p(t) \in V$, then this system is absolutely stable.

Let us assume that in the phase space of parametric perturbations $\{p, v\}$ there exists a closed trajectory such that it begins at an instant t_0 with initial conditions $p(t_0) = p_0$, $v(t_0) = 0$ and ends at an instant t_2 in such a way that $p(t_2) = p(t_0)$, $v(t_2) = v(t_0)$ and $v(t_1) = 0$, $p(t_1) = p_1$ at some instant t_1 ($p_0, p_1 \in [p_-, p_+]$).

Theorem. *In order for system* (29) *to be absolutely stable, it is sufficient that the inequality*

$$\int_{t_0}^{t_1} \mu_{\max}(p, v, w)\, dt + \int_{t_1}^{t_2} \mu_{\max}(p, v, w)\, dt < 0$$

holds when the following estimates are valid

$$\sqrt{\frac{\lambda_{\min}(p(t_0))}{\lambda_{\max}(p(t))}}\, |x(t_0)|\, e^{\frac{1}{2}\int_{t_0}^{t} \mu_{\min}\, d\tau} \leqslant |x(t)| \leqslant \sqrt{\frac{\lambda_{\max}(p(t_0))}{\lambda_{\min}(p(t))}}\, |x(t_0)|\, e^{\frac{1}{2}\int_{t_0}^{t} \mu_{\max}\, d\tau}$$

Here $\mu_{\min}(p,v,w)$ and $\mu_{\max}(p,v,w)$ are the minimal and maximal roots of the equation

$$\det\big(L(p,v,w)-\mu_i H(p)\big)=0,\quad \mu_i=\mu_i(p,v,w),\quad i=1,2,3,4$$

and $\lambda_{\min}(p)$, $\lambda_{\max}(p)$ are the minimal and maximal roots of the equation $\det(H(p)-\lambda E)=0$.

After substituting the coefficients k_i into the above system of equations in variations, this system takes the form

$$\begin{aligned}\ddot z+3\dot z-S_1 z+\frac{S_3}{m}y&=0\\ \ddot y+3\dot y+S_2 y-\frac{S_3}{J_{yy}}z&=0\end{aligned}$$

where

$$\begin{aligned}S_1&=S_1(\dot\varphi,\varphi)=\frac{h_0^2}{l_0^2}\dot\varphi^2+\frac{g\cos^2\sigma_0}{h_0}\cos\varphi-9\\ S_2&=S_2(\ddot\varphi,\dot\varphi,\varphi)=\frac{\mu}{J}\left(mg\cos\varphi-mch_0\dot\varphi^2-J_{yy}\ddot\varphi\right)+\frac{3}{2}\\ S_3&=S_3(\ddot\varphi)=\frac{J_{yy}\cos^2\sigma_0}{h_0}\ddot\varphi\end{aligned}$$

Let us rewrite the matrix A of the system of equations in variation in the Cauchy form:

$$A(\varphi,\dot\varphi,\ddot\varphi)=\begin{pmatrix}0&1&0&0\\ S_1&-9&-\dfrac{S_3}{m}&0\\ 0&0&0&1\\ \dfrac{S_3}{J_{yy}}&0&-S_2&-\dfrac{3}{2}\end{pmatrix}$$

If $\varphi=\text{const}$, then the matrix A takes the form

$$A(\varphi,0,0)=\begin{pmatrix}0&1&0&0\\ P_1&-9&0&0\\ 0&0&0&1\\ 0&0&-P_2&-\dfrac{3}{2}\end{pmatrix}$$

where

$$\begin{aligned}P_1&=P_1(\varphi)=\frac{g\cos^2\sigma_0}{h_0}\cos\varphi-9\\ P_2&=P_2(\varphi)=\frac{\mu}{J}mg\cos\varphi+\frac{3}{2}\end{aligned}$$

According to the theorem given above, we can find the matrix $H(\varphi)$ from the equation

$$A^T(\varphi,0,0)\,H(\varphi)+H(\varphi)\,A(\varphi,0,0)=-E$$

The matrix $L(\varphi,\dot\varphi)$ has the form

$$L=\frac{\partial H}{\partial\varphi}\dot\varphi+A^T(\varphi,\dot\varphi,\ddot\varphi)\,H(\varphi)+H(\varphi)\,A(\varphi,\dot\varphi,\ddot\varphi)$$

Let us apply the above equations to the problem considered in Section 2 of this paper. In order to calculate the elements of the matrices $H(\varphi)$ and $L(\varphi,\dot\varphi)$ and the coefficients of the equations $\det\big(L(\varphi,\dot\varphi)-\mu_i H(\varphi)\big)=0$ and $\det(H(\varphi)-\lambda_i E)=0$, the following parameters of the telescope were taken:

1) the mass of the segment: $m=250$ kg,
2) the length of the segment side: $a=2.4$ m,
3) the height of the center of mass of the segment: $h_0=0.115$ m,
4) the length of the cylinder: $l_0=0.12$ m,

5) the distance between the segment's center of mass A and the points B_1, B_2, B_3: $AB_1 = AB_2 = AB_3 = c = 2$ m,

6) the distance between the point O and the points K_1, K_2, K_3: $d = c + l_0 \cos\sigma_0 = 2.034$ m,

7) the moments of inertia of the segment relative the y-axis: $J_{yy} = \dfrac{5}{24}\, ma^2 = 325.821$ kg m^2.

Figures 5 – 7 illustrate an interesting peculiarity of the solution to this problem, very important for making practical decisions.

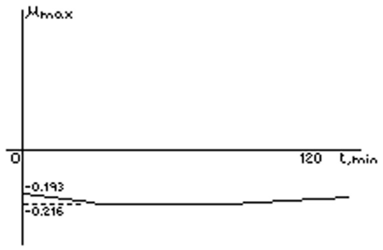

Fig. 5

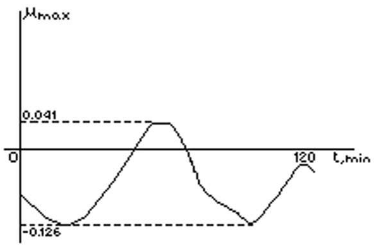

Fig. 6

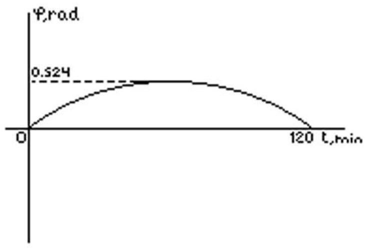

Fig. 7

In the problem under consideration, we have $\varphi \in V = \{0 \leqslant \varphi \leqslant \varphi_+,\ \dot\varphi_- \leqslant \dot\varphi \leqslant \dot\varphi_+,\ 0 \in [\dot\varphi_-, \dot\varphi_+]\}$, where $\dot\varphi_+ = \dfrac{\pi}{6}$ rad, $\dot\varphi_+ = -\dot\varphi_- = \dfrac{\pi}{180}$ rad/s, and φ is a nonperiodic function satisfying the hypotheses of the above theorem.

1) For $\varphi(t) \in V$ the integral is negative and equal to $I = -1494.737$; hence, according to Alexandrov's theorem, the system under investigation is absolutely stable.

2) The estimates obtained give the evidence of a fast (of order e^{-40}) damping of deviations.

3) If the control forces are applied as linear feedback in deviations from programmed lengths of the cylinders, our estimates show that an initial deviation cannot increase more than 7.7 times on a finite time interval.

Thus, a disturbed motion of the segment does not influence the asymptotic stability of relative motion of the segment.

Conclusions. The solution of the above problems of finding optimal parameters of stabilization and the estimates of stabilization precision in the presence of time-varying perturbations ensure the complete solution of the control problem in question. In principle, it is possible to propose another approach based on the problem of robust stabilization in the form of differential games.

REFERENCES

1. K.-M. Lee and D.K. Shah, "Dynamic analysis of a three-degrees-of-freedom-in-parallel actuated manipulator", *IEEE J. of Robotics and Automation*, **4**, 3: 361–367, 1988.
2. D. Stewart, "A platform with six degrees of freedom", *Proc. Inst. Mech. Eng.*, **180**, 15: 371–386, 1965/1966.
3. V.V. Alexandrov, L.I. Voronin, U.N. Glazkov, A.U. Ishlinsky, and V.A. Sadovnichii, *Mathematical Problems of Dynamic Simulation of Aerospace Flights* (in Russian), Moscow University Press, 1995.
4. A.V. Trifonova, *Analysis of Stability of Nonstationary Control Systems*, (in Russian; Thesis), Moscow State University, 1996.
5. L. Guerra, *Calculo de Parametros Optimos Para la Estabiliza de un Segmento del Gran Telscopio Milimetrico* (in Spanish; Thesis), Pueblo University, México, 1996.
6. R.E. Kalman, "Contribution to the theory of optimal control", *Bulletin of the Mexican Mathematics Society*, pp. 102–119, 1960.
7. K. Schittcowski, *Nonlinear Programming Codes: Information, Test, Performance*, Lecture Notes in Economics and Mathematical Systems, Volume 183, Berlin: Springer–Verlag, 1980.