CHAPTER 4

METHODS OF MATHEMATICAL MODELING

SHAPE FIBRATIONS WITH 0-DIMENSIONAL FIBERS

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It is shown that a shape fibration (between metric compacta) with 0-dimensional fibers can be represented as an inverse limit of covering projections of absolute neighborhood retracts (ANRs). This representation is used to establish a lifting theorem for these fibrations in terms of shape groups.

1. Introduction. Covering projections play an important role not only in algebraic topology but in differential geometry, complex analysis, the theory of Lie groups, etc. The central result concerning covering projections is the solution of the following *lifting problem*:

Let $p:(E,e)\to(B,b)$ be a covering projection and $f:(Y,y)\to(B,b)$ be a map. Under what conditions does there exist a *lift* $\bar f:(Y,y)\to(E,e)$, i.e., a map $\bar f$ satisfying $p\circ\bar f=f$?

If Y is connected and locally path-connected, then a necessary and sufficient condition for the lift is the following *lifting criterion*:

$$f_{\#}(\pi_1(Y,y))\subset p_{\#}(\pi_1(E,e))$$

This is the classical *lifting theorem*. This theorem remains true if we take, as a map p, a Hurewicz fibration with a unique path lifting instead of a covering projection (see [10], Chap. 2, Sec. 4, Th. 5, p. 76).

Obviously, the lifting criterion is obtained by applying the fundamental group functor π_1 to the equality $p\circ\bar f=f$. Therefore, this criterion is always a necessary condition for a lift. However, it fails to be sufficient when the space Y is not "locally nice".

That is why the following problem arises: to modify the lifting criterion in order to make it sufficient for a wider class of spaces Y (perhaps, at a sacrifice in reducing the class of maps p). In [4], R. Fox formulated a lifting criterion in terms of first shape groups for the class of all connected metrizable Y, considering a rather special class of covering projections p named *overlays*. Afterwards, in [9] P. Mrozik considerably generalized Fox's lifting theorem, using inverse systems of fundamental groups (*pro-groups*) and improving the definition of an overlay.

In this paper we deal, in fact, with a special class of Hurewicz fibrations with a unique path lifting, namely, with the class of shape fibrations p having 0-dimensional fibers. Thus, the fibers of p may not be necessarily discrete, whereas the fibers of overlays, clearly, are always discrete. Theorems 4.2 and 4.3 are lifting theorems for such kind of fibrations. The corresponding lifting criterion is given in terms of shape and strong shape groups. Although the spaces Y in these theorems still are rather "nice" (pointed one-movable compacta), they need not be locally path connected.

2. Some basic concepts and definitions. In this section, we give all necessary definitions and results for convenience of the reader, although some knowledge of shape theory corresponding to [7] is assumed.

Throughout the paper, we deal only with metric topological spaces and continuous maps. The corresponding category is denoted by $\mathcal{M}$. The notion of shape fibration was first introduced in [8] for metric compacta and was generalized afterwards in [6] for arbitrary spaces. The definition of shape fibration uses the approximate homotopy lifting property (AHLP) applied to ANR-resolutions of maps. We shall need only resolutions for compact metric spaces and for maps of compact metric spaces.

Let $\underline{E}=\{E_i,q_i^j\}$ be an inverse sequence of ANR-spaces (not necessarily compact) and let E be a compactum. A morphism $\underline{q}=\{q_i\}:E\to\underline{E}$ (in pro-$\mathcal{M}$) is an *ANR-resolution* of E if and only if $(E,\underline{q})=\varprojlim\underline{E}$ and the following condition holds: for each i and any neighborhood U of $q_i(E)$ in E_i there exists $j\geqslant i$ such that $g_i^j(E_j)\subseteq U$.

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Let $\underline{p} = \{p_i\} : \underline{E} \to \underline{B}$ (where $p_i : E_i \to B_i$) be a level map of the inverse sequences $\underline{E} = \{E_i, q_i^j\}$ and $\underline{B} = \{B_i, r_i^j\}$. It can be regarded as an inverse sequence $\underline{p} = \{p_i, (q_i^j, r_i^j)\}$ in Map$\mathcal{M}$, where Map$\mathcal{M}$ is the category of maps of $\mathcal{M}$. Let $p : E \to B$ be a map of compact spaces. A morphism $(\underline{q}, \underline{r}) : p \to \underline{p}$ is an *ANR-resolution* of $\underline{p}$ if and only if $\underline{q}$ and $\underline{r}$ are ANR-resolutions of E and B, respectively.

In the case of maps of metric compacta (more generally, in the case of proper maps of metric spaces), shape fibrations can be described using the homotopy lifting property (HLP), which is simpler than the AHLP. Namely, a map $p : E \to B$ of metric compacta is a *shape fibration* if it admits an ANR-resolution $(\underline{q}, \underline{r}) : p \to \underline{p}$ such that for any space X, for each map $h : X \to E_{i+1}$, and for each homotopy $H : X \times I \to B_{i+1}$ with $p_{i+1} \circ h = H_0$ there exists a homotopy $\tilde{H} : X \times I \to E_i$ satisfying $\tilde{H}_0 = q_i^{i+1} \circ h$ and $p_i \tilde{H} = r_i^{i+1} \circ H$.

The lifting property can even more be simplified using the regularity of the HLP and the fact that each map p_i is a composition of an SDR-map $E_i \to \mathrm{coCyl}(p_i)$ and a Hurewicz fibration $\mathrm{coCyl}(p_i) \to B_i$, where the SDR-*map* is a map that embeds one space into another as a strong deformation retract.

Theorem 2.1. *A map $p : E \to B$ of a compact metric space is a shape fibration if and only if p admits an ANR-resolution $p \to \underline{p}$, $\underline{p} = \{p_i\}$, such that each p_i is a Hurewicz fibration.*

For a proof, see [2], Th. 5, p. 203.

A closed subset A of a space X is a *shape strong deformation retract* of X if and only if there exists an absolute retract space M and a closed embedding $\alpha : X \hookrightarrow M$ such that the following condition holds: for each pair of neighborhoods (U, V) of $(\alpha(X), \alpha(A))$ in M there is the homotopy $H : X \times I \to M \,\mathrm{rel.}\, A$ with $H_0 = \alpha$, $\mathrm{Im}\, H \subseteq U$ and $H_1 \subseteq V$.

A map $s : A \hookrightarrow X$ is a SSDR-*map* if and only if s embeds A as a shape strong deformation retract of X.

A closed subset A of a space X is a *strong infinite deformation retract* of X if and only if there exists a *strong infinite deformation of X onto A*, i.e., a map $D : X \times [0, 1) \to X$, such that:

(i) $D(a, t) = a$ if $a \in A$ and $t \in [0, 1)$;

(ii) for any neighborhood U of A in X there exists $\lambda \in [0, 1)$ such that $D(X \times [\lambda, 1)) \subset U$.

Note that if A is a strong infinite deformation retract of X, then $A \hookrightarrow X$ is an SSDR-map.

A space Y is a *fibrant* space if and only if it has the extension property with respect to SSDR-maps in the following sense: for every SSDR-map $s : A \hookrightarrow X$ and every map $A \to Y$ there exists a map $\bar{f} : X \to Y$ such that $\bar{f} \circ s = f$.

Every ANR is a fibrant space. Inverse limits of inverse sequences consisting of fibrants and Hurewicz fibrations as bonding maps are fibrant. The compact metric topological groups are fibrant. The retracts of fibrant spaces are obviously fibrant.

A *fibrant extension* of X consists of a fibrant space $\tilde{X}$ and an SSDR-map $s : X \hookrightarrow \tilde{X}$. Throughout the paper, dealing with fibrant extensions $\tilde{X}$ of X, we always suppose that $X \subseteq \tilde{X}$.

Every compact metric space has a fibrant extension (see [3]).

Each of the ANR-resolutions $\underline{q} : X \to \underline{X}$ (more generally, the resolutions consisting of fibrant spaces) of a compact space X can be used to construct its fibrant extension. Namely, the *cotelescope* $\tilde{X} = \mathrm{coTel}(\underline{X})$ of X (see [5]) with the natural embedding $s : X \hookrightarrow \tilde{X}$ serves as a fibrant extension for X. Moreover, $s(X)$ is a strong infinite deformation retract of $\tilde{X}$.

The cotelescope construction can be realized in the category Map$\mathcal{M}$. In this category, the Hurewicz fibrations of fibrant spaces can be regarded as "fibrant" objects (for more details, see [2]). In a natural way, the concepts of strong deformation and strong infinite deformation are defined in Map$\mathcal{M}$.

For example, *a strong infinite deformation of $\tilde{p}$ onto p* in the category Map$\mathcal{M}$ can be represented by a pair (D, H) of strong infinite deformations

$$D : \tilde{E} \times [0, 1) \to \tilde{E}, \quad H : \tilde{B} \times [0, 1) \to \tilde{B}$$

onto E and B, respectively, such that

$$\tilde{p} \circ D = H \circ (\tilde{p} \times id_{[0,1)})$$

As a consequence of Theorem 2.1 and the cotelescope construction, we get the following

Theorem 2.2 ([2], Th. 9, p. 207). *If $p : E \to B$ is a shape fibration of metric compacta E and B, then there exists a Hurewicz fibration $\tilde{p} : \tilde{E} \to \tilde{B}$ of fibrant extensions $\tilde{E}$ and $\tilde{B}$ of E and B, respectively, such that the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\quad i \quad} & \tilde{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ B & \xrightarrow{\quad j \quad} & \tilde{B} \end{array}$$

Here the inclusions $i : E \hookrightarrow \tilde{E}$ and $j : B \hookrightarrow \tilde{B}$ are SSDR-maps. Moreover, there exists a strong infinite deformation of $\tilde{p}$ onto p in the category $\mathrm{Map}\,\mathcal{M}$.

Let $\underline{q} : X \to \underline{X}$, $\underline{X} = \{X_i, q_i^j\}$, be an ANR-resolution of a compact metric space X. Fixing a point $x \in X$, we can pass on to the "pointed" ANR-resolution $\underline{q} : (X, x) \to (\underline{X}, \underline{x})$, where $\underline{x} = \{x_i\}$ and $x_i = q_i(x)$ for each i.

Let $\underline{G} = \{G_i, f_i^j\}$ be an inverse sequence of groups and homomorphisms. $\underline{G}$ is said to be *Mittag-Leffler* if and only if for each i there exists $j \geqslant i$ such that, for all $k \geqslant j$, $f_i^j(G_j) = f_i^k(G_k)$.

Proposition 2.3 (see [7]). *If $\underline{G} = \{G_i, f_i^j\}$ is Mittag-Leffler, then for each i there exists $j \geqslant i$ such that, for all $k \geqslant j$, $f_i(\varprojlim \underline{G}) = f_i^k(G_k)$, where $f_i : (\varprojlim \underline{G}) \longrightarrow G_i$ are the natural projections.*

Let $\underline{q} : (X, x) \to (\underline{X}, \underline{x})$ be an ANR-resolution of a pointed compactum (X, x).

(X, x) is said to be *pointed one-movable* if and only if $\{\pi_1(X_i, x_i), q^j_{i\#}\}$ is Mittag-Leffler, where $\pi_1(X_i, x_i)$ are fundamental groups and $q^j_{i\#}$ are homomorphisms of fundamental groups induced by the maps q_i^j.

This definition does not depend on the choice of the ANR-resolution $\underline{q} : (X, x) \to (\underline{X}, \underline{x})$. Moreover, if X is connected (i.e., a *continuum*), then it does not depend on the choice of $x \in X$.

Let $\underline{q} : (X, x) \to (\underline{X}, \underline{x})$ be an ANR-resolution of a pointed compact space (X, x). Then, the group $\check{\pi}_1(X, x) = \varprojlim\{\pi_1(X_i, x_i), q^j_{i\#}\}$ is called the first *shape group* of (X, x) (see [7]).

Let $s : X \hookrightarrow \tilde{X}$ be a fibrant extension of a compactum X. The group $\bar{\pi}_1(X, x) = \pi_1(\tilde{X}, s(x))$ is called the first *strong shape group* of (X, x).

The definitions of shape and strong shape groups do not depend on the choice of an ANR-resolution and of a fibrant extension of (X, x), respectively. Moreover, for any map $f : (X, x) \to (Y, y)$ of pointed compact spaces, the homomorphisms $\check{f} : \check{\pi}_1(X, x) \to \check{\pi}_1(Y, y)$ and $\bar{f} : \bar{\pi}_1(X, x) \to \bar{\pi}_1(Y, y)$ are naturally defined in such a way that $\check{\pi}_1$ and $\bar{\pi}_1$ can be considered as functors from the category of compact metric spaces and continuous maps to the category of groups and homomorphisms.

3. 0-dimensional shape fibrations as inverse limits of covering projections. Recall that a topological space X is *0-dimensional* if and only if it has a base consisting of sets that are both open and closed.

For example, all discrete sets and inverse limits of discrete sets are 0-dimensional. The condition of 0-dimensionality formally is the strongest condition of disconnectedness. There are many other conditions of that kind. Here we mention only two of them.

A topological space X is *totally disconnected* if and only if it has only single point components. A topological space X is *totally separated* if and only if for every pair of points $x, y \in X$, $x \neq y$, there exist closed subsets $A, B \subset X$ such that $x \in A$, $y \in B$, $A \cup B = X$, and $A \cap B = \emptyset$.

Every 0-dimensional T_0-space is totally separated. Every totally separated space is totally disconnected. Clearly, in our case of compact metric spaces all these three notions coincide.

A shape fibration $p : E \to B$ is *0-dimensional* if and only if all its fibers are 0-dimensional (in other words, $\dim(p^{-1}(b)) = 0$ for each $b \in B$).

The natural projection $p : D \to S^1$ of the dyadic solenoid D to the circle S^1 is a 0-dimensional shape fibration. Its fibers are homeomorphic to the Cantor set.

Lemma 3.1. *Let $p : \tilde{X} \to X$ be a surjective Hurewicz fibration of ANRs. There exists a commutative diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\quad \gamma \quad} & X^* \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p^*} & \\ & X & \end{array}$$

such that

(a) $X^\star$ is an ANR,

(b) $p^\star$ is a covering projection,

(c) γ is a Hurewicz fibration with path-connected fibers.

Since components of an ANR are open and also are ANRs, without loss of generality we assume that X and $\tilde{X}$ are connected. The map $p^\star$ is constructed as a covering projection corresponding to $p_{\#}(\pi_1(\tilde{X},\tilde{x}))$ for some $\tilde{x} \in \tilde{X}$, whereas the map γ is a lift of p. Note that γ can be considered as a factor map with respect to the partition of $\tilde{X}$ by the following equivalence relation: $\tilde{x}_1 \sim \tilde{x}_2$ if and only if $p(\tilde{x}_1) = p(\tilde{x}_2)$ and, moreover, $\tilde{x}_1, \tilde{x}_2$ can be joined by a path in $p^{-1}(p(\tilde{x}_1))$.

Theorem 3.2. *A map $p : E \longrightarrow B$ between compact metric spaces is a 0-dimensional shape fibration if and only if there exists an ANR-resolution $\underline{p} = \{p_i\} : \underline{E^*} \longrightarrow \underline{B}$ of p, where $p_i : E_i \longrightarrow B_i$ is a covering projection for each i.*

Proof. If p admits an ANR-resolution consisting of covering projections, then, obviously, it is a 0-dimensional shape fibration, since the covering projections are Hurewicz fibrations with 0-dimensional fibers.

Let us assume now that p is a 0-dimensional shape fibration. By Theorem 2.1, there exists an ANR-resolution $\underline{p} = \{p_i\} : \underline{E} \longrightarrow \underline{B}$ of $p : E \longrightarrow B$, $\underline{E} = \{E_i, q_i^j\}$, $\underline{B} = \{B_i, r_i^j\}$, where $p_i : E_i \longrightarrow B_i$ is fibration for each i.

Using Lemma 3.1, we represent every fibration as a composition of a factor map $\gamma_i : E_i \longrightarrow E_i^*$ and a covering projection $p_i^* : E_i^* \longrightarrow B_i$ of ANR-spaces.

For each i, we define $f_i^{i+1} : E_{i+1}^* \longrightarrow E_i^*$ by $f_i^{i+1}(e_{i+1}^*) = \gamma_i \circ q_i^{i+1}(e_{i+1})$ where $e_{i+1}^* \in E_{i+1}^*$ and $e_{i+1} \in \gamma_{i+1}^{-1}(e_{i+1}^*)$. The map f_i^{i+1} is well defined. Indeed, if $e_{i+1}, e_{i+1}^{'} \in \gamma_{i+1}^{-1}(e_{i+1}^*)$, then e_{i+1}, $e_{i+1}^{'}$ are connected by a path $\omega : I \to p_{i+1}^{-1}(p_{i+1}(e_{i+1}))$ and $q_i^{i+1}(e_{i+1})$, $q_i^{i+1}(e_{i+1}^{'})$ are connected by $q_i^{i+1} \circ \omega : I \to p_i^{-1}(p_i \circ q_i^{i+1}(e_{i+1}))$, but this means that $\gamma_i \circ q_i^{i+1}(e_{i+1}) = \gamma_i \circ q_i^{i+1}(e_{i+1}^{'})$. Moreover, we have $f_i^{i+1} \circ \gamma_{i+1} = \gamma_i \circ q_i^{i+1}$. In particular, this implies that f_i^{i+1} is continuous by the definition of factor topology.

Finally, $p_i^* \circ f_i^{i+1} = r_i^{i+1} \circ p_{i+1}^*$. Indeed, if $e_{i+1}^* \in E_{i+1}^*$ and $e_{i+1} \in \gamma_{i+1}^{-1}(e_{i+1}^*)$, then $p_i^* \circ f_{i+1}(e_{i+1}^*) = p_i^* \circ \gamma_i \circ q_i^{i+1}(e_{i+1}) = p_i \circ q_i^{i+1}(e_{i+1}) = r_i^{i+1} \circ p_{i+1}(e_{i+1}) = r_i^{i+1} \circ p_{i+1}^* \circ \gamma_{i+1}(e_{i+1}) = r_i^{i+1} \circ p_{i+1}^*(e_{i+1}^*)$. Therefore, we have the following commutative diagram for each i:

$$\begin{tikzcd}
E_i \arrow[dd, "p_i"'] \arrow[dr, "\gamma_i"] & & E_{i+1} \arrow[ll, "q_i^{i+1}"'] \arrow[dd, "p_{i+1}"'] \arrow[dr, "\gamma_{i+1}"] & \\
& E_i^* \arrow[dl, "p_i^*"'] & & E_{i+1}^*. \arrow[ll, "f_i^{i+1}"'] \arrow[dl, "p_{i+1}^*"] \\
B_i & & B_{i+1} \arrow[ll, "r_i^{i+1}"] &
\end{tikzcd}$$

Thus, we obtained the inverse sequence $\underline{E}^* = \{E_i^*, f_i^{i+1}\}$ of ANR-spaces and the level maps $\underline{p}^* = \{p_i^*\} : \underline{E}^* \longrightarrow \underline{B}$, $\underline{\gamma} = \{\gamma_i\} : \underline{E} \longrightarrow \underline{E}^*$ such that $\underline{p}^* \circ \underline{\gamma} = \underline{p}$.

Now let $p^* = \varprojlim \underline{p}^*$, $E^* = \varprojlim \underline{E}^*$, and $\gamma = \varprojlim \underline{\gamma}$. Then $p^* \circ \gamma = p$.

Let us show that γ is a homeomorphism.

First we prove that γ is injective. Let e_1, $e_2 \in E$ be such that $\gamma(e_1) = \gamma(e_2)$. Hence, $p(e_1) = p(e_2)$ and e_1, $e_2 \in F = p^{-1}(b)$, where $b = p(e_1)$. For each i, the points $e_{1i} = q_i(e_1)$ and $e_{2i} = q_i(e_2)$ can be joined by a path in the fiber $F_i = p_i^{-1}(r_i(b))$, since $\gamma_i(e_{1i}) = \gamma_j(q_i(e_i)) = f_i(\gamma(e_1)) = f_i(\gamma(e_2)) = \gamma_i(q_i(e_2)) = \gamma_i(e_{2i})$.

Note also that $\{F_i, q_i^j|_{F_j}\}$ is an ANR-resolution of F.

Suppose $e_1 \neq e_2$. Since F is 0-dimensional, there exist its closed subsets C_1, C_2 for which $C_1 \cap C_2 = \emptyset$, $C_1 \cup C_2 = F$, and $e_1 \in C_1$, $e_2 \in C_2$. Clear, that $q_i(C_1) \cap q_i(C_2) = \emptyset$ for some i, because C_1 and C_2 are compact. For the closed sets $q_i(C_1)$ and $q_i(C_2)$, we choose open (in E_i) sets $U_1 \supseteq q_i(C_1)$ and $U_2 \supseteq q_i(C_2)$ such that $U_1 \cap U_2 = \emptyset$. Then, the set $U = U_1 \cup U_2$ is an open neighborhood of $q_i(F)$ in E_i and there exists $j \geqslant i$ for which $q_i^j(F_j) \subset U$. The points $e_{1j} = q_j(e_1)$ and $e_{2j} = q_j(e_2)$ can be joined by a path in F_j. Hence, $q_i^j(e_{1j})$ and $q_i^j(e_{2j})$ can be joined by a path in $U = U_1 \cup U_2$. However, this is in contradiction with $U_1 \cap U_2 = \emptyset$.

Now we show that γ is surjective. Let $e^* \in E^*$. We are going to prove the following assertion: for each i, $\gamma_i^{-1}(f_i(e^*)) \cap q_i(E) \neq \emptyset$.

Assume that for some i this is not true. Then, we can find an open neighborhood U of $q_i(E)$ in E_i and some $j \geqslant i$ such that $\gamma_i^{-1}(f_i(e^*)) \cap U = \emptyset$ and $q_i^j(E_j) \subset U$. On the other hand, since γ_j is surjective, there exists $e_j \in E_j$ for which $\gamma_j(e_j) = f_j(e^*)$; therefore, $\gamma_i \circ q_i^j(e_j) = f_i^j \circ \gamma_j(e_j) = f_i^j \circ f_j(e^*) = f_i(e^*)$. Hence, $q_i^j(e_j) \in \gamma_i^{-1}(f_i(e^*))$ and $q_i^j(e_j) \in q_i^j(E_j) \subset U$. However, this is in contradiction with $\gamma_i^{-1}(f_i(e^*)) \cap U = \emptyset$.

The above-proved assertion implies that $e^* \in \overline{\gamma(E)}$, where $\overline{\gamma(E)}$ is the closure of $\gamma(E)$ in E^*. Indeed, if U is a neighborhood of e^* in E^*, then for some i and some neighborhood U_i of $f_i(e^*)$ in E_i^* we have $f_i^{-1}(U_i) \subset U$. If now $e_i \in \gamma_i^{-1}(f_i(e^*)) \cap q_i(E)$, then for some $e \in E$ the equalities $q_i(e) = e_i$ and $f_i(\gamma(e)) = \gamma_i(q_i(e)) = \gamma_i(e_i) = f_i(e^*)$ hold. Hence, $\gamma(e) \in f_i^{-1}(f_i(e^*)) \subset U$.

Since E is compact, $\overline{\gamma(E)} = \gamma(E)$ and we conclude that $e^* \in \gamma(E)$. Thus, $\gamma : E \to E^*$ is a bijection. Hence, it is a homeomorphism, because E is compact. Therefore, we can identify E and E^*, p and p^*.

In order to complete the proof, we should check only that $\underline{p}^*$ is an ANR-resolution of $p^* = p$.

Let U be a neighborhood of $q_i^*(E^*)$ in E_i^*. Then, $V = \gamma_i^{-1}(U)$ is a neighborhood of $q_i(E)$ in E_i. Since $\underline{E}$ is an ANR-resolution, there exists $j \geqslant i$ such that $q_i^j(E_j) \subseteq V$. Recall that γ_i is surjective. Hence,

$$q_i^{*j}(E_j^*) = q_i^{*j} \circ \gamma_j(E_j) = \gamma_i \circ q_i^j(E_j) \subseteq \gamma_i(V) = U$$

Thus, $\underline{p}^*$ is as claimed.

Corollary 3.3. *Let $p : E \to B$ be a 0-dimensional shape fibration between compact metric spaces E and B. Then, p is a Hurewicz fibration with a unique path-lifting property.*

The following statement is a “0-dimensional” version of Theorem 2.2.

Corollary 3.4. *If $p : E \to B$ is a 0-dimensional shape fibration of metric compacta E and B, then there exists a Hurewicz fibration with the unique path lifting $\tilde{p} : \tilde{E} \to \tilde{B}$ of fibrant extensions $\tilde{E}$ and $\tilde{B}$ of E and B, respectively, such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "p"'] & \tilde{E} \arrow[d, "\tilde{p}"] \\ B \arrow[r, "j"] & \tilde{B} \end{tikzcd}$$

Here the inclusions $i : E \hookrightarrow \tilde{E}$ and $j : B \hookrightarrow \tilde{B}$ are SSDR-maps. Moreover, there exists a strong infinite deformation of $\tilde{p}$ onto p in the category Map$\mathcal{M}$.

Clearly, the proof of the corollary is a repetition of the proof of Theorem 2.2 (for the proof, see [2]) with the following modification: for each morphism $\alpha = (q_i^{i+1}, r_i^{i+1}) : p_{i+1} \to p_i$ in the category Map$\mathcal{M}$, where p_i and p_{i+1} are covering projections of ANRs, the cocylinder coCyl(α) is also a covering projection of ANRs and, hence, $\tilde{p} = \mathrm{coTel}(\underline{p})$, being an inverse limit of the cocylinders, is a Hurewicz fibration with the unique path lifting.

Note also that under the conditions of the corollary we have $(\tilde{p})^{-1}(j(B)) = i(E)$, because no point of $(\tilde{p})^{-1}(j(B))$ moves when we carry out the strong infinite deformation.

4. The lifting problem for 0-dimensional shape fibrations.

Lemma 4.1. *Let $p : (E, e) \longrightarrow (B, b)$ be a covering projection of ANR-spaces E and B. If Y is a pointed one-movable continuum, then a map $f : (Y, y) \to (B, b)$ has a (unique) lift $\bar{f} : (Y, y) \to (E, e)$, i.e., a map for which the diagram*

$$\begin{tikzcd} & (E,e) \arrow[d, "p"] \\ (Y,y) \arrow[ur, "\bar{f}"] \arrow[r, "f"] & (B,b) \end{tikzcd}$$

commutes, if and only if f satisfies the lifting criterion: $\check{f}(\check{\pi}_1(Y,y)) \subset p_{\#}(\pi_1(E,e))$.

Proof. It is clear that the lifting criterion is a necessary condition for a lift. We have to prove that it is sufficient.

We may assume that Y is a closed subset of the Hilbert cube Q. There exists a decreasing system $\{Y_i\}_{i\in N}$ of connected ANR-neighborhoods Y in Q such that $\cap_{i=1}^{\infty} Y_i = Y$. In particular, the spaces Y_i are locally path-connected.

Let $q_i^j : (Y_j, y_j) \longrightarrow (Y_i, y_i)$ and $q_i : (Y, y) \longrightarrow (Y_i, y_i)$ be inclusions. Then $\{Y_i, q_i^j\}$ is an ANR-resolution of Y.

Since B is an ANR, there is an extension $h : U \to B$ of the map $f : Y \to B$, where U is some neighborhood of Y in Q. Find $m \in N$ such that $Y_m \subset U$. So, for each $i \geqslant m$ we define a map $f_i : (Y_i, y_i) \longrightarrow (B, b)$ by $f_i = h|_{Y_i}$. Thus, for each $i \geqslant m$, we have $f_i \circ q_i = f$ and $q_i^{i+1} \circ f_{i+1} = f_i$, i.e., the following diagram commutes:

$$\begin{tikzcd}
 & & & & (B,b) \\
(Y_m,y_m) \arrow[urrrr, "f_m"] & (Y_{m+1},y_{m+1}) \arrow[l, "q_m^{m+1}"'] \arrow[urrr, "f_{m+1}"] & (Y_{m+2},y_{m+2}) \arrow[l, "q_{m+1}^{m+2}"'] \arrow[urr, "f_{m+2}"] & \cdots & (Y,y). \arrow[u, "f"']
\end{tikzcd}$$

Let $i \geqslant m$. Since Y is pointed one-movable, by Proposition 2.3 there exists $j \geqslant i$ such that for each $k \geqslant j$ we have $q_{i\#}(\check{\pi}_1(Y,y)) = q_{i\#}^k(\pi_1(Y_k, y_k))$.

Hence, for each $k \geqslant j$ we obtain

$$\check{f}(\check{\pi}_1(Y,y)) = f_{i\#} \circ q_{i\#}(\check{\pi}_1(Y,y)) = f_{i\#} \circ q_{i\#}^k(\pi_1(Y_k, y_k)) = f_{k\#}(\pi_1(Y_k, y_k))$$

Thus, $f_{k\#}(\pi_1(Y_k, y_k)) \subset p_{\#}(\pi_1(E,e))$. By the classical lifting theorem, there exists a unique map $\bar{f}_k$: $(Y_k, y_k) \longrightarrow (E, e)$ such that $p \circ \bar{f}_k = f_k$.

For each $l \geqslant k$, we get $p \circ (\bar{f}_k \circ q_k^l) = (p \circ \bar{f}_k) \circ q_k^l = f_k \circ q_k^l = f_l = p \circ \bar{f}_l$ and hence, by the uniqueness of a lift, we have $\bar{f}_k \circ q_k^l = \bar{f}_l$.

Let $\bar{f} = \varprojlim\{\bar{f}_k\}$. Then, $p \circ \bar{f} = p \circ (\bar{f}_k \circ q_k) = (p \circ \bar{f}_k) \circ q_k = f_k \circ q_k = f$. Thus, $\bar{f}$ is the required lift.

In the proofs of each of the next two lifting theorems, we show only that the corresponding lifting criteria are sufficient. Obviously, they are necessary for a lift, too, since $\check{\pi}_1$ and $\bar{\pi}_1$ are functors.

Theorem 4.2. *Let $p : (E,e) \longrightarrow (B,b)$ be a 0-dimensional shape fibration between compact metric spaces. Let Y be a pointed one-movable continuum. A map $f : (Y,y) \longrightarrow (B,b)$ has a (unique) lift $\bar{f} : (Y,y) \longrightarrow (E,e)$ if and only if $\check{f}(\check{\pi}_1(Y,y)) \subset \check{p}(\check{\pi}_1(E,e))$.*

Proof. Since p is a 0-dimensional shape fibration, by Theorem 3.2 there exists an ANR-resolution $\underline{p} = \{p_i\}$: $\underline{E} \to \underline{B}$ of p, $\underline{E} = \{E_i, q_i^j\}$, $\underline{B} = \{B_i, r_i^j\}$, where for each i the map $p_i : E_i \to B_i$ is a covering projection and $p = \varprojlim\{\underline{p}\}$. In other words, the following diagram commutes:

$$\begin{tikzcd}
(E_1,e_1) \arrow[d, "p_1"'] & (E_2,e_2) \arrow[l, "q_1^2"'] \arrow[d, "p_2"] & \cdots & (E,e) \arrow[d, "p"] \\
(B_1,b_1) & (B_2,b_2) \arrow[l, "r_1^2"] & & (B,b).
\end{tikzcd}$$

For each i we define $f_i : (Y, y) \longrightarrow (B_i, b_i)$ by $f_i = r_i \circ f$, where $r_i : (B, b) \to (B_i, b_i)$ is a natural projection. Since $\check{f}(\check{\pi}_1(Y, y)) \subset \check{p}(\check{\pi}_1(E, e))$ and $f_i = r_i \circ f$, we get $\check{f}_i(\check{\pi}_1(Y, y)) = \check{r}_i \circ \check{f}(\check{\pi}_1(Y, y)) \subset \check{r}_i \circ \check{p}(\check{\pi}_1(E, e)) = p_{i\#} \circ \check{q}_i(\check{\pi}_1(E, e)) \subset p_{i\#}(\pi_1(E_i, e_i))$ and, therefore, $\check{f}_i(\check{\pi}_1(Y, y)) \subset p_{i\#}(\pi_1(E_i, e_i))$.

According to Lemma 4.1, there exists a unique map $\bar{f}_i : (Y, y) \longrightarrow (E_i, e_i)$ such that $p_i \circ \bar{f}_i = f_i$. Let $j \geqslant i$. We have $p_i \circ (q_i^j \circ \bar{f}_j) = (p_i \circ q_i^j) \circ \bar{f}_j = (r_i^j \circ p_j) \circ \bar{f}_j = r_i^j \circ (p_j \circ \bar{f}_j) = r_i^j \circ f_j = f_i = p_i \circ \bar{f}_i$. On the other hand, since $q_i^j \circ f_j(y) = q_i^j(e_i) = e_i = \bar{f}_i(y)$ and $p_i \circ (q_i^j \circ f_j) = p_i \circ \bar{f}_i$, we get $q_i^j \circ \bar{f}_j = \bar{f}_j$ by the uniqueness of a lift. Thus, the family of maps $\{\bar{f}_i\} : Y \longrightarrow \underline{E}$ induces a map $\bar{f} : Y \longrightarrow E$ such that $q_i \circ \bar{f} = \bar{f}_i$ for each i. The map $\bar{f}$ is the desired lift. Indeed, $r_i \circ (p \circ \bar{f}) = p_i \circ (q_i \circ \bar{f}) = p_i \circ \bar{f}_i = f_i$ and $r_i \circ (p \circ \bar{f}) = f_i = r_i \circ f$ for each i. Hence, $p \circ \bar{f} = f$.

Theorem 4.3. *Let $p : (E, e) \longrightarrow (B, b)$ be a 0-dimensional shape fibration between compact metric spaces. Let Y be a pointed one-movable continuum. A map $f : (Y, y) \longrightarrow (B, b)$ has a (unique) lift $\tilde{f} : (Y, y) \longrightarrow (E, e)$ if and only if $\bar{f}(\bar{\pi}_1(Y, y)) \subset \bar{p}(\bar{\pi}_1(E, e))$.*

Proof. By Corollary 3.4, we can consider $p : (E, e) \to (B, b)$ as a restriction of the fibration with the unique path lifting $\tilde{p} : (\tilde{E}, \tilde{e}) \to (\tilde{B}, \tilde{b})$, where $\tilde{E} \supseteq E$ and $\tilde{B} \supseteq B$ are fibrant extensions. Let $\tilde{Y} \supseteq Y$ be a fibrant extension of Y. Since $\tilde{B}$ is a fibrant space, there exists an extension $F : \tilde{Y} \to \tilde{B}$ of the map $j \circ f$, where $j : B \hookrightarrow \tilde{B}$ is an inclusion (in fact, it is an SSDR-map). By the definition of strong shape groups, we have $\bar{f}(\bar{\pi}_1(Y, y)) = F_\#(\pi_1(\tilde{Y}, y))$ and $\bar{p}(\bar{\pi}_1(E, e)) = \tilde{p}_\#(\pi_1(\tilde{E}, e))$ (in $\bar{\pi}_1(B, b) = \pi_1(\tilde{B}, b)$).

Hence, $F_\#(\pi_1(\tilde{Y}, y)) \subset \tilde{p}_\#(\pi_1(\tilde{E}, e))$.

Since Y is the pointed one-movable continuum, $\tilde{Y}$ is a connected locally path-connected space [1]. Therefore, we can apply the classical lifting theorem in order to find a unique lift $\bar{F} : (\tilde{Y}, y) \to (\tilde{E}, e)$ of F, i.e., a map $\bar{F}$ such that $\tilde{p} \circ \bar{F} = F$. Note that $x \in Y$ implies $\bar{F}(x) \in E$, since $\tilde{p}^{-1}(B) = E$. Indeed, $\tilde{p}(\bar{F}(x)) = F(x) = f(x) \in B$.

Clearly, a map $\tilde{f} : (Y, y) \to (E, e)$ defined by $\tilde{f}(x) = \bar{F}(x)$ is the required lift of f.

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