THE PROBLEM OF THE POINTWISE FOURIER INVERSION FOR PIECEWISE SMOOTH FUNCTIONS IN SEVERAL VARIABLES

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In this paper, we study the pointwise convergence $S_{RD}f(\boldsymbol{x}) \to f(\boldsymbol{x})$ of the inverse Fourier transform for piecewise smooth functions in several variables, where $S_{RD}f(\boldsymbol{x}) = \dfrac{1}{(2\pi)^{n/2}} \displaystyle\int\limits_{RD} \hat{f}(\boldsymbol{\xi})\, e^{i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle}\, d\boldsymbol{\xi}$ is the partial sum operator, $\hat{f}(\boldsymbol{\xi}) = \dfrac{1}{(2\pi)^{n/2}} \displaystyle\int\limits_{\mathbb{R}^n} f(\boldsymbol{\xi})\, e^{-i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle}\, d\boldsymbol{x}$ is the Fourier transform of f, D is an open convex domain that contains the point $\boldsymbol{\xi} = \mathbf{0}$, $R > 0$, $RD = \{R\boldsymbol{\xi} : \boldsymbol{\xi} \in D\}$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$, and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

1. Introduction. Let f be a real function of several variables, $\hat{f}$ be its Fourier transform, and

$$S_{RD}f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{RD} \hat{f}(\boldsymbol{\xi})\, e^{i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} d\boldsymbol{\xi} \tag{1.1}$$

where $RD = \{R\boldsymbol{\xi} : \boldsymbol{\xi} \in D\}$ with $R > 0$ and D is an open convex domain with a smooth boundary (the domain D contains the point $\boldsymbol{\xi} = \mathbf{0}$). By the Plancherel theorem, it is known that $\lim\limits_{R\to\infty} S_{RD}f(\boldsymbol{x}) = f(\boldsymbol{x})$ in the L_2-metric. The problem of pointwise convergence of the inverse Fourier transform (or the problem of pointwise inversion) consists in verifying the equality

$$\lim_{R\to\infty} S_{RD}f(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{1.2}$$

for a given point $\boldsymbol{x} \in \operatorname{Dom} f$.

There are piecewise smooth functions such that the limit in (1.2) does not exist. For example (see [5]), if $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, is a ball $B_1(\mathbf{0})$ with its center at the point $\mathbf{0}$, χ_Ω is the characteristic function of Ω, and $D = \Omega = B_1(\mathbf{0})$, then $S_{RD}\chi_\Omega(\boldsymbol{x})$ diverges at $\boldsymbol{x} = \mathbf{0}$.

The problem of pointwise inversion for piecewise smooth functions in several variables has been investigated by Mark A. Pinsky [5, 6]. He studied functions of the type $\chi_\Omega(\boldsymbol{x})\, f(\boldsymbol{x})$, where $f(\boldsymbol{x})$ is a sufficiently smooth function, Ω is an open convex domain with a smooth boundary (Ω contains the point $\mathbf{0}$), and $D = B_1(\mathbf{0})$.

In this paper, we consider a generalization of the same problem for the case when the domains Ω, D, and the function $f(\boldsymbol{x})$ satisfy the conditions listed below.

Condition 1. The domain under consideration is compact, contains the point $\mathbf{0}$, and has a smooth boundary $\partial\Omega$ with nonzero and variable curvature at any of its points.

Condition 2. The function $f(\boldsymbol{x})$ is sufficiently smooth inside Ω and is equal to zero outside Ω (this means that this function has a discontinuity at the boundary $\partial\Omega$).

Suppose S_{RD} is the partial sum operator defined on a domain D which is an open bounded set and satisfies Condition 1, with the exception of compactness. In order to formulate our main results, we first introduce the following definitions.

Definition 1. We say that two domains Ω and D are *Legendre-connected* if there exists $\lambda > 0$ such that the surface $\lambda\,\partial D$ is the Legendre transform of the surface $\partial\Omega$.

Definition 2. We say that two domains Ω and D are *in general position* if there exists $\lambda > 0$ such that the surface $\lambda\,\partial D$ is tangent to the Legendre transform $L\Omega$ of the surface $\partial\Omega$ and this tangency is of general position, i.e., these surfaces do not osculate.

Our main results are as follows.

Assertion A. *If Ω and D are Legendre-connected, then the limit in* (1.2) *does not exist.*

Assertion B. *If Ω and D are in general position, then there is $\varepsilon > 0$ such that, for $\boldsymbol{x} \in B_\varepsilon(\mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < \varepsilon\}$, $\lim\limits_{R\to\infty} S_{RD}f(\boldsymbol{x})$ exists and is equal to $f(\boldsymbol{x})$.*

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In order to prove these assertions, we should find an asymptotic representation of $\hat{f}(\boldsymbol{\xi})$ for $\|\boldsymbol{\xi}\| \gg 1$ and, then, use this representation to calculate the limit in (1.2).

2. An asymptotic representation of $\hat{f}(\boldsymbol{\xi})$ for $\|\boldsymbol{\xi}\| \gg 1$. Let Ω and f satisfy Conditions 1 and 2. Let $\hat{f}(\boldsymbol{\xi})$ stand for the Fourier transform of the function f, i.e.,

$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\Omega} f(\boldsymbol{x})\, e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}\, d\boldsymbol{x} \tag{2.1}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

In order to find an asymptotic representation of $\hat{f}(\boldsymbol{\xi})$ as $\|\boldsymbol{\xi}\| \to \infty$, we introduce the following notation: let us rewrite $\boldsymbol{\xi}$ in the form

$$\boldsymbol{\xi} = k\,\boldsymbol{\omega}$$

where $k = \|\boldsymbol{\xi}\| \geqslant 0$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$, and $\|\boldsymbol{\omega}\| = 1$. In terms of this notation, we shall seek the asymptotic representation of $\hat{f}(\boldsymbol{\xi})$ for a fixed $\boldsymbol{\omega}$ and $k \to \infty$.

Let $(y_1, y_2, y_3, \ldots, y_{n-1}, y_n)$ be orthogonal coordinates[2] in $\mathbb{R}^n$ such that

$$y_1 = \langle \boldsymbol{\omega}, \boldsymbol{x}\rangle = x_1\omega_1 + x_2\omega_2 + \cdots + x_{n-1}\omega_{n-1} + x_n\omega_n$$

Then, $\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle = k\, y_1$ and

$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{M_-}^{M_+} e^{-iky_1}\, \Phi(y_1)\, dy_1 \tag{2.2}$$

where

$$\Phi(y_1) = \int\limits_{(y_1, y_2, \ldots, y_n)\in\Omega} f(y_1, y_2, \ldots, y_n)\, dy_2\, dy_3 \ldots dy_n \tag{2.3}$$

$$M_+ = \max_{x\in\Omega} \langle \boldsymbol{\omega}, \boldsymbol{x}\rangle = \max_{(y_1, y_2, \ldots, y_n)\in\Omega} \{y_1\} \quad \text{and} \quad M_- = \min_{x\in\Omega} \langle \boldsymbol{\omega}, \boldsymbol{x}\rangle = \min_{(y_1, y_2, \ldots, y_n)\in\Omega} \{y_1\} \tag{2.4}$$

Since Ω satisfies Condition 1, the boundary $\partial\Omega$ is strictly convex. Therefore, there exist only two points P_+ and P_- at $\partial\Omega$ for which the points M_+ and M_- are their projections on the y_1-axis.

The asymptotic behavior of the integral in (2.2) is specified by singularities of $\Phi(y_1)$. Taking into account these singularities, we can formulate the following lemma.

Lemma 1. *The function $\Phi(y_1)$ is equal to zero in $(-\infty, M_-) \cup (M_+, \infty)$; there exists $\varepsilon > 0$ such that for $0 < M_+ - y_1 < \varepsilon$ and for $0 < y_1 - M_- < \varepsilon$ this function has the series expansions*

$$\Phi_{P_+}(y_1) = \frac{2^{\frac{n-1}{2}}}{\sqrt{\left|\hat{C}(P_+)\right|}} (M_+ - y_1)^{\frac{n-1}{2}} \left\{ \left[f(P_+)V_{n-1} + r_0'\right] + \sum_{j=1}^{N-1} r_j' \,(M_+ - y_1)^j \right\} + O\,(M_+ - y_1)^{\frac{n-1}{2}+N} \tag{2.5}$$

and

$$\Phi_{P_-}(y_1) = \frac{2^{\frac{n-1}{2}}}{\sqrt{\left|\hat{C}(P_-)\right|}} (y_1 - M_-)^{\frac{n-1}{2}} \left\{ \left[f(P_-)\, V_{n-1} + q_0'\right] + \sum_{j=1}^{N-1} q_j' \,(y_1 - M_-)^j \right\} + O\,(y_1 - M_-)^{\frac{n-1}{2}+N} \tag{2.6}$$

where $N > 1$ depends on smoothness (on the number of continuous derivatives) of the function f and of the surface $\partial\Omega$, V_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$, $\hat{C}(P_\pm)$ are the Gaussian curvatures of $\partial\Omega$ at the points $P_\pm$, and r_j', q_j' are constants.

It follows from (2.2) that $\hat{f}(\boldsymbol{\xi})$ is the Fourier transform in the variable y_1 of the function $\Phi(y_1)$. In accordance with the standard methods of stationary phase (see, for example, [2, 4]), expansions (2.5), (2.6) of $\Phi(y_1)$ lead to the following theorem on the asymptotic behavior of $\hat{f}(\boldsymbol{\xi})$.

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[2] To define these coordinates, we first complete the orthonormal basis $\{\boldsymbol{\omega}, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \ldots, \boldsymbol{\omega}_n\}$ and next denote $y_i = \langle \boldsymbol{\omega}_i, \boldsymbol{x}\rangle$.

Theorem 1. *The asymptotic representation of* $\hat{f}(\boldsymbol{\xi})$ *(as* $k=\|\boldsymbol{\xi}\|\to\infty$*) is*

$$\begin{aligned}\hat{f}(\boldsymbol{\xi}) \cong{}& \frac{(i)^{\frac{n+1}{2}}\, e^{-ikM_{+}}\Gamma(\frac{n+1}{2})}{\sqrt{2}\,\pi^{n/2}\sqrt{\left|\hat{C}(P_{+})\right|}\,k^{\frac{n+1}{2}}}\left[f(P_{+})V_{n-1}+r_{0}+\sum_{j=1}^{N-1} r_{j}\left(\frac{i}{k}\right)^{j}\right]\\ &+\frac{(-i)^{\frac{n+1}{2}}\, e^{-ikM_{-}}\Gamma(\frac{n+1}{2})}{\sqrt{2}\,\pi^{n/2}\sqrt{\left|\hat{C}(P_{-})\right|}\,k^{\frac{n+1}{2}}}\left[f(P_{-})V_{n-1}+q_{0}+\sum_{j=1}^{N-1} q_{j}\left(-\frac{i}{k}\right)^{j}\right]\end{aligned}\tag{2.7}$$

where r_j, q_j *are constants and* $(\pm i)^m=\exp\left(\pm im\,\dfrac{\pi}{2}\right)$.

3. Pointwise inversion of the Fourier transform of f. In this section, we shall use asymptotic representation (2.7) to study the problem of pointwise inversion of the Fourier transform of f.

Let us choose $R_0>0$ such that (2.7) is applicable outside R_0D. Then, for $R>R_0$ we obtain

$$\begin{aligned}(2\pi)^{n/2}S_{RD}f(\boldsymbol{x}) &= \int\limits_{RD}\hat{f}(\boldsymbol{\xi})\,e^{i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}\,d\boldsymbol{\xi}=\int\limits_{R_0D}\hat{f}(\boldsymbol{\xi})\,e^{i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}\,d\boldsymbol{\xi}+\int\limits_{RD\setminus R_0D}\hat{f}(\boldsymbol{\xi})\,e^{i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}\,d\boldsymbol{\xi}\\ &\cong\int\limits_{R_0D}\hat{f}(\boldsymbol{\xi})\,e^{i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}\,d\boldsymbol{\xi}+I(R_0,R,\boldsymbol{x})\end{aligned}\tag{3.1}$$

Here

$$I(R_0,R,\boldsymbol{x})=\int\limits_{RD\setminus R_0D} g(\boldsymbol{\xi})\,e^{i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}\,d\boldsymbol{\xi}\tag{3.2}$$

where $g(\boldsymbol{\xi})$ is given by (2.7).

It is proved below that for a sufficiently small $\|\boldsymbol{x}\|$ a limit of $I(R_0,R,\boldsymbol{x})$ exists as $R\to\infty$. Therefore, $\lim\limits_{R\to\infty} S_{RD}f(\boldsymbol{x})$ exists and is equal to $f(\boldsymbol{x})$.

Let Ω be the domain defined above. For $\boldsymbol{\xi}\in\mathbb{R}^n$, let

$$\Phi_{+}(\boldsymbol{\xi})=\max_{x\in\Omega}\langle\boldsymbol{x},\boldsymbol{\xi}\rangle\,,\quad \Phi_{-}(\boldsymbol{\xi})=\min_{x\in\Omega}\langle\boldsymbol{x},\boldsymbol{\xi}\rangle\tag{3.3}$$

Since $k=\|\boldsymbol{\xi}\|$, from the definitions of M_{+} and M_{-} we conclude that

$$\Phi_{+}(\boldsymbol{\xi})=\|\boldsymbol{\xi}\|\,M_{+}(\boldsymbol{\xi})\quad\text{and}\quad\Phi_{-}(\boldsymbol{\xi})=\|\boldsymbol{\xi}\|\,M_{-}(\boldsymbol{\xi})\tag{3.4}$$

3.1. Proof of Assertion A. Let the boundary ∂D be described by the equation $\boldsymbol{\xi}=\rho(\boldsymbol{\omega})\,\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the unit vector. Obviously, $\rho(\boldsymbol{\omega})$ is the distance between the point $\mathbf{0}$ and the boundary ∂D in the direction $\boldsymbol{\omega}$. A parameter ν is introduced in order to define the "quasipolar" coordinates $(\nu,\boldsymbol{\omega})$ as follows:

$$\boldsymbol{\xi}(\nu,\boldsymbol{\omega})=\nu\rho(\boldsymbol{\omega})\,\boldsymbol{\omega}\tag{3.5}$$

Definition. Let T_1 and T_2 be convex domains with boundaries S_1 and S_2, respectively. We say that S_2 is the Legendre transform of S_1 if $\langle\boldsymbol{x},\boldsymbol{y}\rangle\leqslant 1$ for any $\boldsymbol{x}\in T_1$, $\boldsymbol{y}\in T_2$ and if for any $\boldsymbol{x}\in S_1$ there exists $\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x})\in S_2$ such that $\langle\boldsymbol{x},\boldsymbol{y}\rangle=1$.

Lemma 2. *Let* $R_0>0$ *be such that the above asymptotic representation of* $\hat{f}(\boldsymbol{\xi})$ *is valid for* $\nu>R_0$ *and let* D *be the domain considered above. Then*

$$I(R_0,R,\boldsymbol{x})=\int\limits_{R_0}^{R}\nu^{\frac{n-3}{2}}\,\Psi(\nu,\boldsymbol{x})\,d\nu\tag{3.6}$$

Here

$$\Psi(\nu,\boldsymbol{x})\cong\int\limits_{\boldsymbol{\gamma}\in\partial D}\|\boldsymbol{\gamma}\|^{-\frac{(n-1)}{2}}\cos\Theta(\boldsymbol{\gamma})\left[Ae^{-i\nu[\Phi_{+}(\boldsymbol{\gamma})-\langle\boldsymbol{x},\boldsymbol{\gamma}\rangle]}+Be^{-i\nu[\Phi_{-}(\boldsymbol{\gamma})-\langle\boldsymbol{x},\boldsymbol{\gamma}\rangle]}\right]d\boldsymbol{\gamma}$$

$$A=\frac{f(P_{+})\,(i)^{\frac{n+1}{2}}}{2^{n/2}\sqrt{\pi}\sqrt{\left|\hat{C}(P_{+})\right|}},\quad B=\frac{f(P_{-})\,(-i)^{\frac{n+1}{2}}}{2^{n/2}\sqrt{\pi}\sqrt{\left|\hat{C}(P_{-})\right|}}$$

where $\Theta(\boldsymbol{\gamma})$ is the angle between the vector $\boldsymbol{\omega}$ and the normal to the surface ∂D at the point $\boldsymbol{\gamma}$ and $d\boldsymbol{\gamma}$ is the surface measure on ∂D.

The convergence of the improper integral $\int\limits_{R_0}^{\infty} \nu^{\frac{n-3}{2}} \Psi(\nu, \mathbf{0})\, d\nu$ depends on the behavior of the function $\Psi(\nu, \mathbf{0})$. This function can be such that this integral diverges. For example, suppose $\Phi_{\pm}(\boldsymbol{\gamma})$ are constants. In accordance with (3.3) and (3.4), this means that the domains Ω and D are Legendre-connected. Letting $\Phi_{\pm}(\gamma) = \alpha_{\pm}$, we get

$$\Psi(\nu, \mathbf{0}) \cong C_{-} e^{-i\alpha_{-}\nu} + C_{+} e^{-i\alpha_{+}\nu}$$

where $C_{\pm}$ are some constants. Thus, substituting this relation into the right-hand side of (3.6), we obtain the following integrals:

$$C_{\pm} \int\limits_{R_0}^{R} e^{-i\alpha_{\pm}\nu} \nu^{\frac{n-3}{2}}\, d\nu$$

This integral obviously diverges as $R \to \infty$. This proves Assertion A.

3.2. Proof of Assertion B. In order to find an asymptotic representation of $\Psi(\nu, \mathbf{0})$ as $\nu \to \infty$, we write down

$$\Psi(\nu, \mathbf{0}) = \Psi_1(\nu, \mathbf{0}) + \Psi_2(\nu, \mathbf{0})$$

where

$$\Psi_1(\nu, \mathbf{0}) = A \int\limits_{\boldsymbol{\gamma} \in \partial D} \|\gamma\|^{-\frac{(n-1)}{2}} \cos\Theta(\boldsymbol{\gamma})\, e^{-i\nu\Phi_{+}(\boldsymbol{\gamma})}\, d\boldsymbol{\gamma} \tag{3.7}$$

and

$$\Psi_2(\nu, \mathbf{0}) = B \int\limits_{\boldsymbol{\gamma} \in \partial D} \|\gamma\|^{-\frac{(n-1)}{2}} \cos\Theta(\boldsymbol{\gamma})\, e^{-i\nu\Phi_{-}(\boldsymbol{\gamma})}\, d\boldsymbol{\gamma} \tag{3.8}$$

Now, let us find the asymptotic representations of $\Psi_1(\nu, \mathbf{0})$ and $\Psi_2(\nu, \mathbf{0})$.

The stationary points $\boldsymbol{\gamma}_j$ of the restriction to ∂D of the function $\Phi_{+}(\boldsymbol{\gamma})$ are the points at which the surfaces of constant level ($\Phi_{+} = \text{const}$) are tangent to ∂D. It follows from (3.3) and (3.4) that these surfaces are Legendre-connected with the domain Ω. The condition of general position of the domains Ω and D means that all the critical points $\boldsymbol{\gamma}_j$ are nondegenerate. Therefore, we can use the standard multidimensional method of stationary phase (see, for example, [3, 4]) to obtain the asymptotic representation of $\Psi_1(\nu, \mathbf{0})$

$$\Psi_1(\nu, \mathbf{0}) \cong \sum_{j=0}^{s} \frac{A_j^{\mathbf{0}}\, e^{-i\nu\Phi_{+}(\boldsymbol{\gamma}_j)}}{\nu^{\frac{n-1}{2}}}$$

where the sum is taken over all the stationary points of $\Phi_{+}(\boldsymbol{\gamma})$ and $A_j^{\mathbf{0}}$ are constants that depend, in particular, on the values of the second derivatives of $\Phi_{+}(\boldsymbol{\gamma})$ at the stationary points $\boldsymbol{\gamma}_j$. In a similar way, we can obtain the asymptotic representation of $\Psi_2(\nu, \boldsymbol{\omega})$. Based on these results, we come to the following lemma.

Lemma 3. *The asymptotic representation of $\Psi(\nu, \mathbf{0})$ (as $\nu \to \infty$) is*

$$\Psi(\nu, \mathbf{0}) \cong \sum_{j=0}^{s} \left[A_j^{\mathbf{0}} \frac{e^{-i\nu\Phi_{+}(\boldsymbol{\gamma}_j)}}{\nu^{\frac{n-1}{2}}} + B_j^{\mathbf{0}} \frac{e^{-i\nu\Phi_{-}(\boldsymbol{\gamma}_j)}}{\nu^{\frac{n-1}{2}}} \right]$$

where $\boldsymbol{\gamma}_j$ are the stationary points of Φ_{+} and Φ_{-} at the surface ∂D and $A_j^{\mathbf{0}}$ and $B_j^{\mathbf{0}}$ are constants.

It follows from this lemma that

$$I(R_0, R, \mathbf{0}) \cong \sum_{j=0}^{s} \left[A_j^{\mathbf{0}} \int\limits_{R_0}^{R} \frac{e^{-i\nu\Phi_{+}(\boldsymbol{\gamma}_j)}}{\nu}\, d\nu + B_j^{\mathbf{0}} \int\limits_{R_0}^{R} \frac{e^{-i\nu\Phi_{-}(\boldsymbol{\gamma}_j)}}{\nu}\, d\nu \right]$$

Evidently, these integrals converge as $R \to \infty$. Therefore, $\lim\limits_{R\to\infty} S_{RD} f(\mathbf{0})$ exists. We denote this limit by $\tilde{f}(\mathbf{0})$.

Now, let us prove that $\tilde{f}(\mathbf{0}) = f(\mathbf{0})$.

There exists $\varepsilon > 0$ such that if $\|\boldsymbol{x}\| < \varepsilon$, then $\pounds_+(\boldsymbol{x},\boldsymbol{\gamma}) = \Phi_+(\boldsymbol{\gamma}) - \langle \boldsymbol{x},\boldsymbol{\gamma}\rangle$ and $\pounds_-(\boldsymbol{x},\boldsymbol{\gamma}) = \Phi_-(\boldsymbol{\gamma}) - \langle \boldsymbol{x},\boldsymbol{\gamma}\rangle$ have the same number of stationary points $\boldsymbol{\gamma}_j^{\boldsymbol{x}}$ as Φ_+ and Φ_- (here $\boldsymbol{\gamma} \in \partial D$ and $\boldsymbol{x} \in \mathbb{R}^n$). These points are nondegenerate; hence, there is $\delta = \delta(\varepsilon)$ such that there exists one and only one point $\boldsymbol{\gamma}_j^{\boldsymbol{x}}$ in the δ-neighborhood of an appropriate point $\boldsymbol{\gamma}_j$. Evidently, $\boldsymbol{\gamma}_j^{\boldsymbol{x}}$ and $\pounds_+(\boldsymbol{x},\boldsymbol{\gamma}_j^{\boldsymbol{x}})$ are continuous functions of $\boldsymbol{x}$. The following lemma specifies the sought-for asymptotic representation of $\Psi(\nu,\boldsymbol{x})$.

Lemma 4. *Let $\varepsilon > 0$ be defined above and $\boldsymbol{x} \in B_\varepsilon(\mathbf{0})$; then the asymptotic representation of $\Psi(\nu,\boldsymbol{x})$ (as $\nu \to \infty$) is*

$$\Psi(\nu,\boldsymbol{x}) \cong \sum_{j=0}^{s} \left[A_j^{\boldsymbol{x}} \frac{e^{-i\nu \pounds_+(\boldsymbol{\gamma}_j^{\boldsymbol{x}})}}{\nu^{\frac{n-1}{2}}} + B_j^{\boldsymbol{x}} \frac{e^{-i\nu \pounds_-(\boldsymbol{\gamma}_j^{\boldsymbol{x}})}}{\nu^{\frac{n-1}{2}}} \right]$$

where $A_j^{\boldsymbol{x}}$ and $B_j^{\boldsymbol{x}}$ are continuous functions of $\boldsymbol{x}$.

It follows from this lemma that $I(R_0,R,\boldsymbol{x}) = \int\limits_{R_0}^{R} \nu^{\frac{n-3}{2}}\Psi(\nu,\boldsymbol{x})\,d\nu$ converges for $\boldsymbol{x} \in B_\varepsilon(\mathbf{0})$ and, therefore, the limit of $S_{RD}f(\boldsymbol{x})$ exists as $R \to \infty$. Denote this limit by $\widetilde{f}(\boldsymbol{x})$ (note that this function is continuous in $\boldsymbol{x}$ on the domain $B_\varepsilon(\mathbf{0})$). Assertion B follows from

Lemma 5. *For $\boldsymbol{x} \in B_\varepsilon(\mathbf{0})$, the equality $\widetilde{f}(\boldsymbol{x}) = f(\boldsymbol{x})$ holds.*

In order to prove this lemma, we recall that $S_{RD}f$ converges to f in the L_2-metric. Therefore,

$$\lim_{R\to\infty} \int\limits_{B_\varepsilon(\mathbf{0})} |S_{RD}f - f|^2\,d\boldsymbol{x} = 0$$

Because $\lim\limits_{R\to\infty} S_{RD}f(\boldsymbol{x}) = \widetilde{f}(\boldsymbol{x})$, we can conclude that

$$\int\limits_{B_\varepsilon(\mathbf{0})} \left|\widetilde{f}(\boldsymbol{x}) - f(\boldsymbol{x})\right|^2 d\boldsymbol{x} = 0$$

Since $\widetilde{f}(\boldsymbol{x}) - f(\boldsymbol{x})$ is continuous in $B_\varepsilon(\mathbf{0})$, $\widetilde{f}(\boldsymbol{x}) = f(\boldsymbol{x})$ for any $\boldsymbol{x} \in B_\varepsilon(\mathbf{0})$. Thus, Assertion B is proved.

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