MATHEMATICAL AND NUMERICAL ANALYSIS OF PERIODIC SOLUTIONS NEAR STATIONARY STATES IN A NONLINEAR MODEL OF BRAIN CORTEX ACTIVATION

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A nonlinear ordinary differential equation is considered as a macroscopic (global) model of brain cortex activation. The nonlinearity has the form of a sigmoid function of two parameters a and b. The zoning of the plane (b, a) with respect to the number and type of stationary states (critical points) for the model under consideration is performed. The existence of the center-type solutions in a particular domain of the plane (b, a) is proved analytically. A numerical algorithm for finding periodic solutions is proposed; this algorithm was realized with the MATLAB system and was verified using several model examples.

1. Introduction. The model of brain cortex activation presented in the form of a nonlinear ordinary differential equation was first formulated by P. Ñũnez [1] as a result of analysis done on a system of two integral equations that describe a space-time evolution of synaptic activity of the brain cortex. This model was considered in [2] under some assumptions concerning anatomical and functional characteristics of the brain cortex (such as its columnar organization, the distribution of synaptic connections, etc.) and was based on the use of the *activation variable* $u = u(y, \tau)$. The authors of [2] proposed the following differential equation describing the above model:

$$\frac{\partial^2 u}{\partial \tau^2} + 2\,\frac{\partial u}{\partial \tau} + u - \frac{\partial^2 u}{\partial y^2} = \left[g(u) + \frac{\partial g(u)}{\partial \tau} \right] \tag{1}$$

Here y and τ are the space and time variables, respectively. The function $g(u)$ (called a *shot function*) is expressed by

$$g(u, a, b) = \frac{1}{1 + a\exp(-bu)} \tag{2}$$

where the parameters a and b are positive. This function represents a fraction of the total number of neurons of a column that shoot the action potentials over a period of time t. Its behavior is sigmoidal [2], well known in problems of this type. All variables and parameters of this model are dimensionless.

2. The equation of stationary state. Analysis of critical points. In order to obtain an equation describing the stationary states, it is necessary to eliminate the time-dependent terms entering into (1). As a result, we obtain the equation

$$\frac{d^2 u}{dy^2} + g(u, a, b) - u = 0 \tag{3}$$

Let us consider its periodic solutions corresponding to the boundary conditions $u(0) = u(l)$ and $u'(0) = u'(l)$, where l is a period.

2.1. Analysis of stationary states. Equation (3) can be transformed to the following Hamiltonian system of differential equations:

$$\begin{aligned} \frac{du}{dy} &= z \\ \frac{dz}{dy} &= -F(u) \end{aligned} \tag{4}$$

Here the function $F(u) = u - g(u, a, b)$ is auxiliary. The Hamiltonian (or the function of total energy) is of the form [3]

$$H(u, z) = T(z) + U(u) \tag{5}$$

where $T(z) = \dfrac{z^2}{2}$ is the kinetic energy and $U(u)$ is the potential energy expressed by

$$U(u) = -\int_0^u F(s)\,ds = \frac{1}{b}\ln\frac{\exp(bu) + a}{1 + a} - \frac{u^2}{2} \tag{6}$$

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The stationary states of the above Hamiltonian system are critical points that, in their turn, correspond to the extreme points of the function $U(u)$ [3]. Hence, the critical points can be determined from the equation $U'(u) = 0$ that leads to $g(u, a, b) = u$, i.e., to the equation

$$\frac{1}{1 + a\exp(-bu)} = u \tag{7}$$

The following cases of graphical representations of solutions to equation (7) are given in Figure 1,A for different values of a and b: 1) the solid line corresponds to the case when a single critical point exists; 2) the dashed line illustrates the case of two critical points, and 3) the broken line corresponds to the case of three critical points.

Let us determine the domains on a plane (b, a) where the above cases are realized. The case of two critical points follows from equation (7) and the condition of tangency $g'(u) = 1$:

$$\begin{gathered} \frac{1}{1 + a\exp(-bu)} = u \\ \frac{ab\exp(-bu)}{\left(1 + a\exp(-bu)\right)^2} = 1 \end{gathered} \tag{8}$$

Solving this system, we obtain the equation

$$bu^2 - bu + 1 = 0 \tag{9}$$

whose solution yields the tangency points

$$u^0_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4}{b}} \tag{10}$$

Since $D = 1 - \dfrac{4}{b}$ should be positive, we get the inequality $b > 4$. Substituting (10) into the first equation in (8), we come to the following dependence between a and b:

$$a_\pm = \left(\frac{b}{2}\left(1 \mp \sqrt{1 - \frac{4}{b}}\right) - 1\right)\exp\left(\frac{b}{2}\left(1 \pm \sqrt{1 - \frac{4}{b}}\right)\right)$$

This dependence is represented in Figure 1,B. The curves corresponding to a_+ and a_- divide the plane into three domains Ω_k $(k = 1, 2, 3)$ as follows: 1) the domains Ω_1 and Ω_2 contain the values of a and b for the case of a single critical point, 2) their boundaries correspond to the case of two critical points, and 3) the domain Ω_3 presents the case of three critical points.

2.2. Conditions for stability of the critical points. The center-type stable critical points correspond to a minimum of the function $U(u)$ [3]. We can see from Figure 1,B that these points are situated only in the domain Ω_3 (i.e., between the curves a_+ and a_-). The function $U(u)$ has three extrema when $(b, a) \in \Omega_3$. Only one of these extrema is a local minimum. Note that this minimum corresponds to the center-type case and satisfies the inequality $U''(u) = -F'(u) > 0$, which, together with equation (7), yields the condition for the existence of stable critical points:

$$\begin{gathered} \frac{1}{1 + a\exp(-bu)} = u \\ \frac{ab\exp(-bu)}{\left(1 + a\exp(-bu)\right)^2} > 1 \end{gathered} \tag{11}$$

Solving (11), we obtain the inequality

$$bu^2 - bu + 1 < 0 \tag{12}$$

that is valid for $b > 4$ and $u^0_- < u < u^0_+$, where $u^0_\pm$ are determined by (10). Expressing the above conditions of stability in terms of the parameters a and b, we come to the conclusion that Ω_3 is the domain of stability. Thus, we can formulate the following theorem.

Theorem. *Ordinary differential equation* (3) *has a center-type critical point* u^0 *corresponding to given values* b^0 *and* a^0 *of the parameters* b *and* a *if and only if the conditions* $(b^0, a^0) \in \Omega_3$ *and* $u^0_- < u^0 < u^0_+$ *hold. Here* $u^0_\pm$ *are determined by expression* (10).

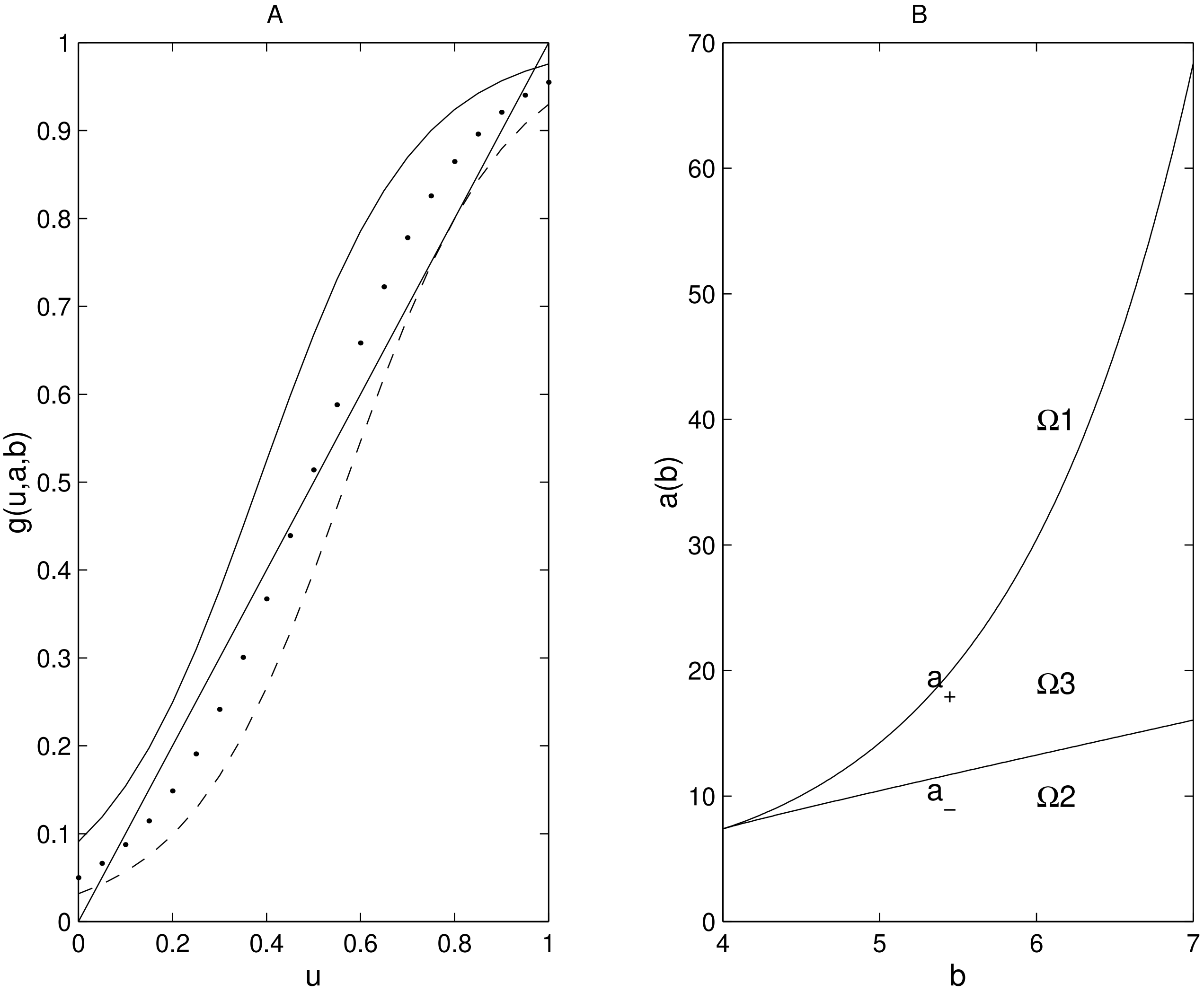

Fig. 1. Analysis of critical points. A: the number of critical points for different values of a and b.
B: the domains with the different number of critical points

3. Numerical results. Let us assume that u_0 is a center-type critical point of equation (3). Taken its solution in the form $u = \overline{u} + u_0$, from (3) we obtain the equation

$$\frac{d^2\overline{u}}{dy^2} + \left(\frac{1}{1 + a\exp(-b\overline{u})\exp(-bu_0)} - \frac{1}{1 + a\exp(-bu_0)}\right) - \overline{u} = 0 \tag{13}$$

Now we consider the boundary conditions $\overline{u}\,(0) = \overline{u}\,(l)$, $\overline{u}\,'\,(0) = \overline{u}\,'\,(l)$ (where l is an unknown period) and construct the corresponding periodic solutions to equation (13). Note that near the center-type critical point all such solutions can be represented by closed (periodic) trajectories on the phase plane. Hence, the boundary value problem can be reduced to the Cauchy problem for equation (13) with the initial conditions

$$\overline{u}(0) = \overline{u}_0, \quad \overline{u}\,'(0) = \overline{u}_0{}' \tag{14}$$

where $\overline{u}_0$ and $\overline{u}_0{}'$ are some given values. In order to solve problem (13), (14) numerically, we introduce a uniform grid $\{y_i\}_{i=1}^{N}$ with a step length h and approximate the derivatives by the corresponding difference formulas. As a result, we obtain a system of nonlinear algebraic equations, which can be solved by the least-squares method. The above algorithm was realized with the MATLAB system, and some model examples were solved numerically.

The periodic solutions obtained near the center $u_0 = 0.472$ are represented in Figure 2 for the following cases: $b = 6$, $a = 19$ ($b, a \in \Omega_3$), $\overline{u}_0 = -0.01$, $\overline{u}_0{}' = 0.01$ (the solid line), and $\overline{u}_0 = -0.03$, $\overline{u}_0{}' = 0.01$ (the dashed

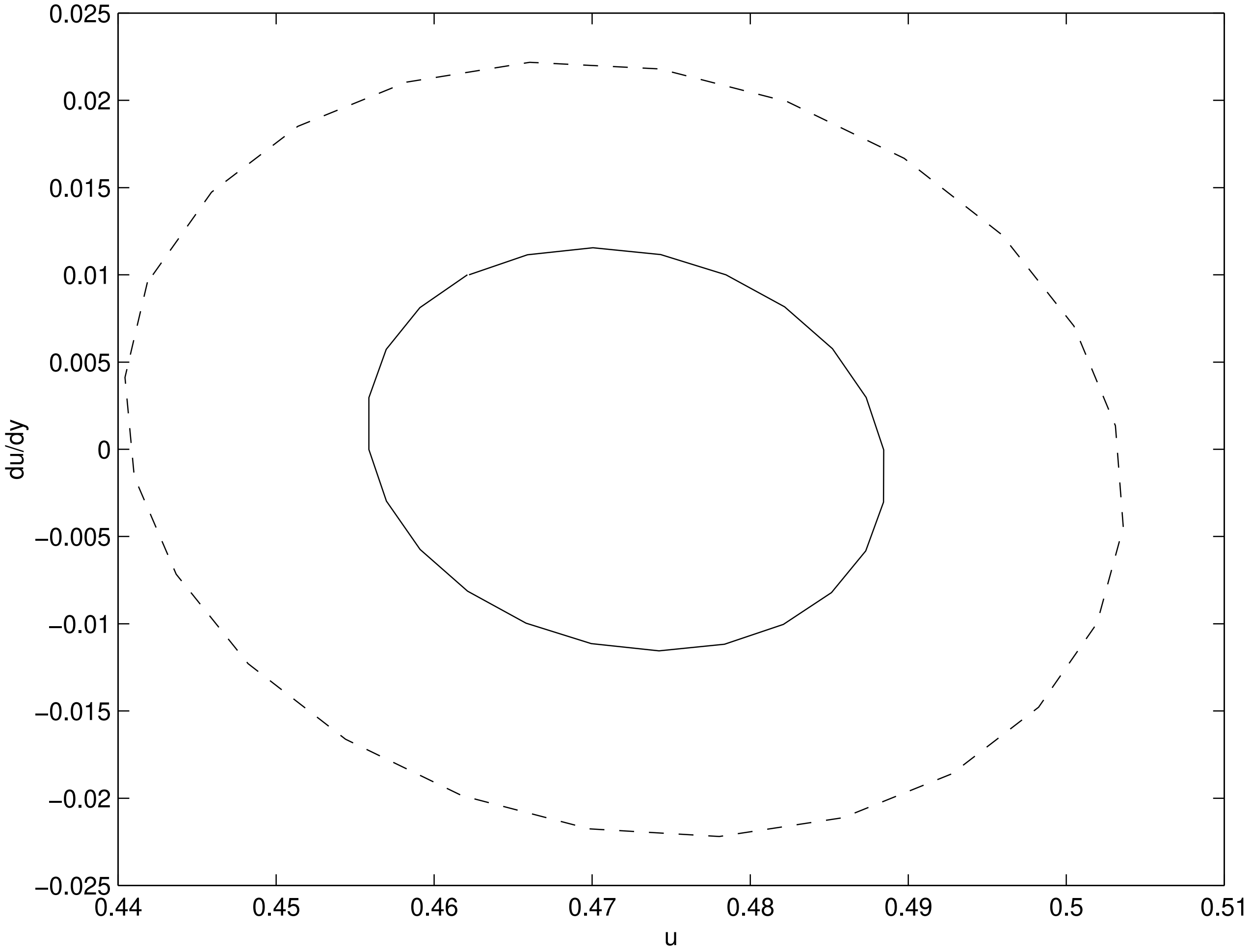

Fig. 2. The behavior of periodic solutions near the center

line). The numerical experiments we performed confirm our theoretical conclusions. The above results can be useful for further studies of basic rhythms of the brain cortex.

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