STABILIZATION OF OSCILLATORY BILINEAR SYSTEMS

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The problem of stabilization of oscillatory bilinear systems with n degrees of freedom is considered. It is supposed that each degree of freedom corresponds to an oscillation frequency and that the system is under the action of additive and parametric controls. It is shown that if all the frequencies are unequal or only two of them coincide, then it is possible to stabilize the system; otherwise, the system is not stabilizable.

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1. The case of two degrees of freedom. Let us consider the controllable bilinear oscillatory system with two degrees of freedom:

$$\begin{cases} \ddot{x}_i + (\omega_i^2 + u_1)\, x_i = u_2\,, \quad i = 1, 2 \\ u_i(\cdot) \in U_i = \{u_i(\cdot) \in KC^1 |\, |u_i(t) \leqslant \nu_i < \infty\} \\ \omega_i^2 - \nu_1 > 0 \end{cases} \tag{1}$$

We shall use the following piecewise linear functions as controls:

$$u_i = \nu_i \widetilde{\operatorname{sign}}\, s_i = \begin{cases} \dfrac{\nu_i}{\delta}\, s_i & \text{for} \quad |s_i| \leqslant \delta \\ \nu_i \operatorname{sign} s_i & \text{for} \quad |s_i| > \delta \end{cases}$$

Mathematical model (1) describes, for example, relative oscillations of two disconnected pendulums when the suspension "segment" moves with the vertical u_1 and horizontal u_2 accelerations ($\omega_i^2 = \dfrac{g_0}{l_i}$ and $\varphi_i = \dfrac{x_i}{l_i}$ are the deviations from the vertical position, l_i is the length of the ith pendulum, and g_0 is the gravitational acceleration).

It is required to determine controls u_1 and u_2 such that the trivial solution of system (1) is asymptotically stable; in other words, it is necessary to stabilize system (1).

First we assume that $\omega_1 \neq \omega_2$. In this case, the additive control u_2 is sufficient for the stabilization. Assuming $u_1 \equiv 0$, therefore, we can rewrite (1) in matrix form:

$$\ddot{x} + \Omega x = \mathbf{1} u_2 \tag{2}$$

Here $\Omega = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$ and $\mathbf{1}^T = (1,\, 1)$.

System (2) is completely controllable [1]; hence, it can be stabilized. The simplest stabilization control has the form

$$s_2 = -\mathbf{1}^T \dot{x}$$

Indeed, in this case the Lyapunov function $v_1 = \dfrac{1}{2}\left(\dot{x}^T \dot{x} + x^T \Omega x\right)$ is infinitely large and its derivative has the form

$$\frac{dv_1}{dt} = -\frac{\nu_x}{\delta}\left(\mathbf{1}^T \dot{x}\right)^2$$

by virtue of system (2). The Barbashin–Krasovskii theorem is applicable, since $\omega_1 \neq \omega_2$; hence, the trivial solution is asymptotically stable.

Now we suppose that $\omega_1 = \omega_2 = \omega_0$. In this case, system (2) is not controllable; moreover, after decomposition the uncontrollable subsystem takes the form $\ddot{\xi}_1 + \omega_0^2 \xi_1 = 0$, i.e., it is not stabilizable. Hence, we cannot stabilize system (2). For the stabilization we should use the multiplicative (parametric) control u_1. Since the

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two-dimensional bisector plane $x_1 = x_2$, $x_3 = x_4$ ($x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$) can be regarded as a controllable subspace for system (2), it is natural to consider the auxiliary system

$$\ddot{x}_0 + \left(\omega_0^2 + u_1(\cdot)\right) x_0 = 0 \tag{3}$$

for the new coordinate $x_0 = x_1 - x_2$ and the extremal problem

$$x_1(t_1) \to \min_{u_1(\cdot)} \tag{4}$$

on minimization of the oscillation amplitude of this system for the limiting case $\delta = 0$. Here $x_1(0) = -1$, $\dot{x}_1(0) = \dot{x}_1(t_1) = 0$, $\dot{x}_1(t_1) \neq 0$ $\forall t \in (0, t_1)$.

Using the Pontrjagin maximum principle, we can prove that problem (3), (4) has the only solution with the optimal control

$$u_1^0 = \nu_1 \operatorname{sign}(x_0 \dot{x}_0)$$

If $u_1 = \nu_1 \widetilde{\operatorname{sign}}(x_0 \dot{x}_0)$, then the asymptotic stability of auxiliary system (3) is easily proved, since we can use the Barbashin–Krasovskii theorem for the Lyapunov function $v_2 = \dfrac{1}{2}\left(x_0^2 + \dot{x}_0^2\right)$, which has the derivative $\dfrac{dv_2}{d\tau} = -\dfrac{\nu_1}{\omega_0 \delta}\, x_0^2\, \dot{x}_0^2 \leqslant 0$ by virtue of system (3); here $\tau = \omega_0 t$.

Thus, we can consider the following stabilization law for system (1):

$$u_1 = \nu_1 \widetilde{\operatorname{sign}}\big[(x_1 - x_2)(\dot{x}_1 - \dot{x}_2)\big], \quad u_2 = -\nu_2 \widetilde{\operatorname{sign}}(\dot{x}_1 + \dot{x}_2)$$

On the segments of linear stabilization, closed system (1) takes the form

$$\begin{aligned} \ddot{x}_1 + \big[\omega_0^2 + k_1(x_1 - x_2)(\dot{x}_1 - \dot{x}_2)\big] x_1 &= -k_2(\dot{x}_1 + \dot{x}_2) \\ \ddot{x}_2 + \big[\omega_0^2 + k_1(x_1 - x_2)(\dot{x}_1 - \dot{x}_2)\big] x_2 &= -k_2(\dot{x}_1 + \dot{x}_2) \end{aligned} \tag{5}$$

where $k_i = \dfrac{v_i}{\delta}\, s_i$, $s_1 = (x_1 - x_2)(\dot{x}_1 - \dot{x}_2)$, $s_2 = -\dot{x}_1 - \dot{x}_2$. The characteristic equation in variations written for system (5) is as follows:

$$\lambda^4 + 2k_2\lambda^3 + 2\omega_0^2\lambda^2 + 2k_2\omega_0^2\lambda + \omega_0^4 = 0 \tag{6}$$

Equation (6) has the two pure imaginary roots $\lambda_{1,2} = \pm i\omega_0$ and the two roots with negative real parts:

$$\lambda_{3,4} = \begin{cases} -k_2 \pm \sqrt{k_2^2 - \omega_0^2} & \text{if} \quad \omega_0 < k_2 \\ -k_2 & \text{if} \quad \omega_0 = k_2 \\ -k_2 \pm i\sqrt{\omega_0^2 - k_2^2} & \text{if} \quad \omega_0 > k_2 \end{cases}$$

Therefore, in order to prove the asymptotic stability in this case, we should use the linear nondegenerate transformation $x = Sy$, where

$$S = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & \omega_0 & 0 & 1 \\ 0 & -\omega_0 & 0 & 1 \end{pmatrix}$$

In these new coordinates, system (5) has the form

$$\begin{cases} \dot{y}_1 = \omega_0 y_2 \\ \dot{y}_2 = -\omega_0 y_1 - k_1 y_1^2 y_2 \end{cases} \tag{7}$$

$$\begin{cases} \dot{y}_3 = y_4 \\ \dot{y}_4 = -\omega_0 y_3 - 2k_2 y_4 - k_1\omega_0 y_1 y_2 y_3 \end{cases} \tag{8}$$

Subsystem (7) coincides with the auxiliary system if $u_1 = \nu_1 \widetilde{\operatorname{sign}}(x_0 \dot{x}_0)$; its trivial solution, therefore, is asymptotically stable. Subsystem (8) has the following matrix form:

$$\dot{y} = \big(A + B(t)\big) y$$

Here $A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2k_2 \end{pmatrix}$ is the Hurwitz matrix and $B = \begin{pmatrix} 0 & -0 \\ -k_1\omega_0 y_1(t)y_2(t) & 0 \end{pmatrix}$. All the elements of the matrix B tend to zero asymptotically for arbitrary initial conditions. Therefore, if we use the Bellman–Granwall lemma, then we can show that all the solutions of subsystem (8) tend to zero asymptotically as well. Thus, the trivial solution of system (5) is asymptotically stable, i.e., the problem on stabilization of original system (1) is solved. If we have measuring instruments that register precise information on all the coordinates x_1, x_2, $\dot{x}_1$, $\dot{x}_2$, then we can construct a closed system whose mathematical model is of the form

$$\ddot{x}_1 + \left[\omega_0^2 + \nu_1 \widetilde{\operatorname{sign}}\big((x_1 - x_2)(\dot{x}_1 - \dot{x}_2)\big)\right] x_1 = -\nu_2 \widetilde{\operatorname{sign}}(\dot{x}_1 + \dot{x}_2)$$
$$\ddot{x}_2 + \left[\omega_0^2 + \nu_1 \widetilde{\operatorname{sign}}\big((x_1 - x_2)(\dot{x}_1 - \dot{x}_2)\big)\right] x_2 = -\nu_2 \widetilde{\operatorname{sign}}(\dot{x}_1 + \dot{x}_2)$$

If we combine the cases $\omega_1 \neq \omega_2$ and $\omega_1 = \omega_2$, then we can obtain a complete solution to the problem on stabilization of system (1).

2. The case of n degrees of freedom ($n > 2$). Let us consider the controllable bilinear oscillatory system with n degrees of freedom ($n > 2$):

$$\begin{cases} \ddot{x}_i + \left(\omega_i^2 + u_1\right) x_i = u_2, \;\; i = 1, \ldots, n \\ u_j(\cdot) \in \mathbb{U} = \left\{u_j(\cdot) \in KC \mid |u_j| \leqslant \nu_j < \infty, \;\; j = 1, 2\right\} \\ \omega_i^2 - \nu_1 > 0 \end{cases} \tag{9}$$

Our aim is to formulate conditions under which the trivial solution of system (9) is asymptotically stable. As before, the system is under the action of the parametric control u_1 and the additive control u_2.

2.1. The case of unequal frequencies. Let us consider the case when the parametric control is absent and all the frequencies are distinct. Then, the following lemma is valid:

Lemma [1]. *The controllable system*

$$\begin{cases} \ddot{x} + \Omega x = \mathbf{1} u_2 \\ u_2 = K^T \dot{x} \\ \Omega = \begin{pmatrix} \omega_1^2 & 0 & 0 \ldots & 0 \\ 0 & \omega_2^2 & 0 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & \omega_n^2 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \end{cases} \tag{10}$$

is stabilizable when $\omega_i^2 \neq \omega_j^2 \;\; \forall\, i \neq j,\; i, j = 1, \ldots, n$, i.e., there exists an $(n \times 1)$-matrix K such that the trivial solution to the equation $\ddot{x} + \Omega x = K^T \dot{x}$ is asymptotically stable.

Since the controls under consideration are bounded, the next theorem formulates the condition when the problem on stabilization of system (10) can be solved:

Theorem. *If system* (10) *is subjected to the action of the control*

$$u_2 = -\nu_2 \widetilde{\operatorname{sign}}\left(\mathbf{1}^T \dot{x}\right) = \begin{cases} -\dfrac{\nu_2}{\kappa} \mathbf{1}^T \dot{x} & \text{if } \left|\mathbf{1}^T \dot{x}\right| \leqslant \kappa \\ -\nu_2 \operatorname{sign}\left(\mathbf{1}^T \dot{x}\right) & \text{if } \left|\mathbf{1}^T \dot{x}\right| \geqslant \kappa \end{cases}$$

then its trivial solution is asymptotically stable.

2.2. The case when two frequencies coincide. Let us consider system (9) for the case when $\omega_1^2 = \omega_2^2 = \omega_0^2 \neq \omega_j^2$, $j = 3, 4, \ldots, n$, and $\omega_i^2 \neq \omega_j^2 \;\; \forall\, i \neq j,\; i, j = 3, \ldots, n$.

As was shown above in the case $n = 2$, for the stabilization we should use a parametric control u_1. In order to apply this approach to system (9) for $n > 2$, we rewrite (9) in the form

$$\begin{cases} \ddot{x}_1 + (\omega_0^2 + u_1)\, x_1 = u_2 \\ \ddot{x}_2 + (\omega_0^2 + u_1)\, x_2 = u_2 \\ \ddot{x}_3 + (\omega_3^2 + u_1)\, x_3 = u_2 \\ \vdots \\ \ddot{x}_n + (\omega_n^2 + u_1)\, x_n = u_2 \\ u_i(\cdot) \in \mathbb{U} = \left\{u_i \in KC \;\mid |u_i(t)| \leqslant \nu_i < \infty, \; i = 1, 2\right\} \end{cases}$$

Introducing the new coordinate $x_0 = x_1 - x_2$, we can decompose the above system into the two subsystems

$$\begin{aligned}&(\mathrm{I}) \quad \left\{\ddot{x}_0 + (\omega_0^2 + u_1)\, x_0 = 0\right.\\ &(\mathrm{II}) \quad \begin{cases} \ddot{x}_2 + (\omega_0^2 + u_1)\, x_2 = u_2 \\ \ddot{x}_3 + (\omega_3^2 + u_1)\, x_3 = u_2 \\ \vdots \\ \ddot{x}_n + (\omega_n^2 + u_1)\, x_n = u_2 \end{cases}\end{aligned} \tag{11}$$

The following theorem holds:

Theorem. *System* (11) *is locally stabilizable if the controls*

$$u_1 = \nu_1 \widetilde{\mathrm{sign}}\left((x_1 - x_2)(\dot{x}_1 - \dot{x}_2)\right)$$
$$u_2 = -\nu_2 \widetilde{\mathrm{sign}}\,(\dot{x}_2 + \dot{x}_3 + \cdots + \dot{x}_n)$$

are used.

2.3. The case when three or more frequencies coincide. In order to prove that under this condition the system is not stabilizable, we first consider the case when $n = 3$ and the parametric control is absent. Denoting $\omega_1^2 = \omega_2^2 = \omega_3^2 = \omega_0^2$, we rewrite the original system in the form

$$\begin{cases} \ddot{x}_1 + \omega_0^2 x_1 = u_2 \\ \ddot{x}_2 + \omega_0^2 x_2 = u_2 \\ \ddot{x}_3 + \omega_0^2 x_3 = u_2 \end{cases} \tag{12}$$

This system can be written in matrix form as

$$\dot{x} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -w_0^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w_0^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -w_0^2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} u_2$$

The controllability matrix takes the form

$$C = \begin{pmatrix} 0 & 1 & 0 & -w_0^2 & 0 & -w_0^4 \\ 0 & 1 & 0 & -w_0^2 & 0 & -w_0^4 \\ 0 & 1 & 0 & -w_0^2 & 0 & -w_0^4 \\ 1 & 0 & -w_0^2 & 0 & -w_0^4 & 0 \\ 1 & 0 & -w_0^2 & 0 & -w_0^4 & 0 \\ 1 & 0 & -w_0^2 & 0 & -w_0^4 & 0 \end{pmatrix}$$

The first two columns of this matrix specify the controllable subspace for system (12). Hence, we can use the change of coordinates

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix}$$

to decompose system (12) into the following two subsystems:

$$(\mathrm{I}) \quad \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\omega_0^2 y_2 \\ \dot{y}_3 = y_4 \\ \dot{y}_4 = -\omega_0^2 y_3 \end{cases}$$

$$(\mathrm{II}) \quad \begin{cases} \dot{y}_5 = y_6 \\ \dot{y}_6 = -\omega_0^2 y_5 + u_2 \end{cases}$$

It can be concluded from here that subsystem (II) is completely controllable, whereas subsystem (I) is not controllable.

The same reasonings can be used in the case $n > 3$ when at least three frequencies coincide. First, the system is decomposed with a transformation similar to that considered above and, then, it is demonstrated that no stabilization can be reached with the help of any parametric control.

Theorem. *Let us consider the bilinear system with n degrees of freedom* $(n > 3)$

$$\begin{cases} \ddot{x}_1 + (\omega_0^2 + u_1)\, x_1 = u_2 \\ \ddot{x}_2 + (\omega_0^2 + u_1)\, x_2 = u_2 \\ \ddot{x}_3 + (\omega_0^2 + u_1)\, x_3 = u_2 \\ \ddot{x}_4 + (\omega_4^2 + u_1)\, x_4 = u_2 \\ \vdots \\ \ddot{x}_n + (\omega_n^2 + u_1)\, x_n = u_2 \end{cases}$$

and with three equal frequencies. Then the system is not stabilizable with any controls u_1 and u_2.

Proof. By introducing the new coordinates $z_1 = x_1 - x_3$ and $z_2 = x_2 - x_3$, the above system is decomposed into the two subsystems:

$$\text{(I)} \quad \begin{cases} \ddot{z}_1 + (\omega_0^2 + u_1)\, z_1 = 0 \\ \ddot{z}_2 + (\omega_0^2 + u_1)\, z_2 = 0 \end{cases}$$

$$\text{(II)} \quad \begin{cases} \ddot{x}_3 + (\omega_0^2 + u_1)\, x_3 = u_2 \\ \ddot{x}_4 + (\omega_4^2 + u_1)\, x_4 = u_2 \\ \vdots \\ \ddot{x}_n + (\omega_n^2 + u_1)\, x_n = u_2 \end{cases}$$

In the subspace for system (I), we define the function

$$\varphi(t) = z_1 \dot{z}_2 - z_2 \dot{z}_1$$

The time derivative of this function is equal to zero:

$$\frac{d\varphi(t)}{dt} \equiv 0$$

Hence, $\varphi(t) \equiv \text{const}$. Using the polar coordinate system, we can write

$$z_1 = R_1(t) \cos \theta_1(t), \quad \dot{z}_1 = R_1(t) \sin \theta_1(t)$$
$$z_2 = R_2(t) \cos \theta_2(t), \quad \dot{z}_2 = R_2(t) \sin \theta_2(t)$$

and

$$\varphi(t) = R_1(t) R_2(t) \sin\bigl(\theta_2(t) - \theta_1(t)\bigr) \equiv \text{const}$$

Therefore, $R_1(t)R_2(t) \geqslant \text{const}$ for any u_1. Hence, in the above subspace it is impossible to stabilize the system. The theorem is proved.

REFERENCES

1. V.V. Alexandrov, S.I. Zlochevskii, S.S. Lemak, and N.A. Parusnikov, *Introduction to Dynamics of Control Systems* (in Russian), Moscow, 1993.